SIAM J. NUMER. ANAL.
Vol. 38, No. 6, pp. 1961–1985

ADAPTIVE GALERKIN METHODS WITH ERROR CONTROL FOR A DYNAMICAL GINZBURG–LANDAU MODEL IN SUPERCONDUCTIVITY*

ZHIMING CHEN† AND SHIBIN DAI‡

Dedicated to Professor Karl-Heinz Hoffmann on the occasion of his 60th birthday

Abstract. The time-dependent Ginzburg–Landau model which describes the phase transitions taking place in superconductors is a coupled system of nonlinear parabolic equations. It is discretized semi-implicitly in time and in space via continuous piecewise linear finite elements. A posteriori error estimates are derived for the $L^\infty L^2$ norm by studying a dual problem of the linearization of the original system, other than the dual of error equations. Numerical simulations are included which illustrate the reliability of the estimators and the flexibility of the proposed adaptive method.

Key words. a posteriori error estimates, Ginzburg–Landau vortices, superconductivity, adaptive, nonlinear PDEs

AMS subject classifications. 65M15, 65M60, 82D55

PII. S0036142998349102

1. Introduction.

The phenomenological Ginzburg–Landau complex superconductivity model is designed to describe the phenomenon of vortex structure in the superconducting/normal phase transitions. The time-dependent Ginzburg–Landau (TDGL) model derived by Gor'kov and Éliashberg in [17] from averaging the microscopic Bardeen–Cooper–Schrieffer (BCS) theory offers a useful starting point in studying the dynamics of superconductivity. Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary $\partial\Omega$ and let $(0,T)$ be the time interval. Denote by $Q_T = \Omega\times(0,T)$ and $\Gamma_T = \partial\Omega\times(0,T)$. After proper nondimensionalization, the TDGL model can be formulated as in the following system of PDEs:

$$\eta\partial_t\psi + \mathbf{i}\eta\kappa\phi\psi + \Big(\frac{\mathbf{i}}{\kappa}\nabla + \mathbf{A}\Big)^2\psi + (|\psi|^2-1)\psi = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\partial_t\mathbf{A} + \nabla\phi + \mathbf{curl}\,\mathrm{curl}\,\mathbf{A} + \Re\Big[\Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\bar\psi\Big] = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$\Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\cdot\mathbf{n} = 0, \quad \mathrm{curl}\,\mathbf{A} = H \quad \text{on } \Gamma_T, \tag{1.3}$$

$$\psi(\cdot,0) = \psi_0(\cdot), \quad \mathbf{A}(\cdot,0) = \mathbf{A}_0(\cdot) \quad \text{on } \Omega, \tag{1.4}$$

where $\mathbf{n} = (n_1,n_2)$ denotes the exterior unit normal of the boundary $\partial\Omega$, $\Re[\,\cdot\,]$ denotes the real part of the quantity in the brackets $[\,\cdot\,]$, and curl, **curl** denote the curl operators on $\mathbf{R}^2$ defined by

$$\mathrm{curl}\,\mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \quad \mathbf{curl}\,u = \Big(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\Big)^T.$$

*Received by the editors November 25, 1998; accepted for publication (in revised form) August 30, 2000; published electronically January 25, 2001. The authors were partially supported by China NSF grant 19771080 and China National Key Project "Large Scale Scientific and Engineering Computing."
http://www.siam.org/journals/sinum/38-6/34910.html

†Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China. Current address: LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing 100080, People's Republic of China (zmchen@lsec.cc.ac.cn).

‡Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China. Current address: Department of Mathematics, University of Maryland, College Park, MD 20742 (sdai@math.umd.edu).

Here ψ is a complex valued function and is usually referred to as the order parameter so that $|\psi|^2$ gives the relative density of the superconducting electron pairs, and the normal and the pure superconducting states are characterized accordingly by $|\psi|^2 = 0$ and $|\psi|^2 = 1$. $\bar{\psi}$ stands for the complex conjugate of ψ. $\mathbf{A}$ is a real vector potential for the total magnetic field and ϕ is a real scalar function called electric potential. H is the applied magnetic field, viewed as a vector, the magnetic field points out of the (x_1, x_2)-plane. η, κ are positive constants which are related to the known physical quantities.

Global existence of unique strong solutions is obtained in [8] for the TDGL model under the Lorentz gauge

$$\phi = -\mathrm{div}\,\mathbf{A} \quad \text{in } Q_T \quad \text{and} \quad \mathbf{A}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_T. \tag{1.5}$$

With this gauge choice, a semi-implicit finite element scheme is proposed and analyzed in [6]. A mixed finite element method is studied in [5] which approximates the potential $\mathbf{A}$, the magnetic field $\mathrm{curl}\,\mathbf{A}$, and the electric potential $\varphi = \mathrm{div}\,\mathbf{A}$ simultaneously. We also refer to [11], [12], and [22] for the analytical and numerical studies of the TDGL model under other gauge choices.

It is known that for the type-II superconductors, in which case the Ginzburg–Landau constant $\kappa > 1/\sqrt{2}$, the superconducting property is destroyed in some isolated points (two dimensions) or isolated curves (three dimensions) in the mixed state. In the two dimensional case, these isolated points are called vortex points or vortices. The understanding of the properties of vortices such as creation and annihilation, pinning, and nucleation is one of the most important issues in the study of superconductivity. Our previous numerical experiences indicate that usually fine finite element meshes are required in order to resolve the vortex structures. The purpose of this paper is to explore the possibility of adaptively controlling finite element meshes and time-steps. For more readings on the physics of superconductivity we refer to [4], [13], and [26].

A posteriori error estimates are computable quantities that measure the actual errors without knowledge of the limit solution. They are essential in designing algorithms for mesh and time-step modification which equidistribute the computational effort and therefore optimize the computations. Since the seminal paper [2] on elliptic problems, there has been ever increasing interest in the development of reliable and efficient adaptive algorithms for various linear and nonlinear PDEs. In particular, a posteriori error estimates have been derived in [14], [15] for linear and mildly nonlinear parabolic problems and in [25], [9] for degenerate parabolic problems of Stefan type with or without convection. The main tool in deriving a posteriori error estimates in [14], [15], [25], [9] is the analysis of linear dual problems of the corresponding *error* equations. This method has been extended recently in [19] to the TDGL model under gauge choice (1.5). However, we observe that, in this situation, the resulting dual problem has discrete solutions in the coefficients, and, consequently, the proof of strong stability estimates similar to those used in [14] for linear parabolic problems requires further condition on the uniform boundness of the gradient of discrete solutions [19, Proposition 5.7].

In this paper we introduce and analyze a new method to derive a posteriori estimates for the TDGL model which provides the necessary information to modify the mesh and time-step according to the varying external magnetic field and corresponding motion of vortices. The estimates, which exhibit the same characteristics as that of linear parabolic equations [14], are based on the analysis of a dual problem which is the dual of the linearization of the original TDGL system with gauge choice (1.5), other than the error equations. We show that the additional terms in error representation formula due to the change of dual problem is of higher order and thus can be absorbed under suitable nondegeneracy assumption (see section 5 for details). We remark that the nondegeneracy assumption, which is not very restrictive in practices, is used in [1], [23], [24] to obtain both the upper and lower bound of a posteriori error estimates for linear elliptic problems. The simulations in section 6 clearly show the reliability and flexibility of the adaptive algorithm based on our a posteriori error estimators. We finally remark that the method proposed in the present paper to derive a posteriori error estimates can be extended to other nonlinear parabolic equations with smooth nonlinearities.

The paper is organized as follows. In section 2 we state the notation and set the problem. In section 3 we discuss the semi-implicit finite element scheme. In section 4 we introduce the parabolic dual problem and prove the strong stability estimates. In section 5 we prove the a posteriori error estimates. Finally in section 6 we show the performance of the adaptive finite element methods based on our estimators.

2. Setting. We first introduce some of the notation to be used in the paper. If X denotes some Banach space of real scalar functions, the corresponding space of complex scalar functions will be denoted by its calligraphic form $\mathcal{X}$ and the corresponding space of real vector-valued functions, each of its components belonging to X, will be denoted by its boldfaced form $\mathbf{X}$. However, we shall use $\|\cdot\|_X$ to denote the norms of the Banach spaces X, $\mathcal{X}$, or $\mathbf{X}$. We shall also use the subspace

$$\mathbf{H}_n^1(\Omega) = \left\{ \mathbf{B} \in \mathbf{H}^1(\Omega) : \ \mathbf{B}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

It is known that the following embedding inequality holds on $\mathbf{H}_n^1(\Omega)$ (cf. [16]):

$$\|\mathbf{B}\|_{H^1(\Omega)} \le C\Big[\|\mathbf{B}\|_{L^2(\Omega)} + \|\mathrm{div}\,\mathbf{B}\|_{L^2(\Omega)} + \|\mathrm{curl}\,\mathbf{B}\|_{L^2(\Omega)}\Big] \quad \forall \mathbf{B} \in \mathbf{H}_n^1(\Omega), \tag{2.1}$$

where the constant C depends on the domain Ω.

Now we state the hypotheses concerning the data.

(H1) $\psi_0 \in \mathcal{H}^2(\Omega), \mathbf{A}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_n^1(\Omega)$ satisfying $|\psi_0| \le 1$ on Ω;
(H2) $H \in H^1(0,T;H^{1/2}(\partial\Omega))$.

In view of (H2) we may consider H extended to Ω in such a way that $H \in H^1(0,T;H^1(\Omega))$. In what follows we assume for convenience that

(H3) given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $H_{\text{ext}} \in H^1(\Omega)$, the linear elliptic problem

$$-\Delta\mathbf{Q} = \mathbf{f} \ \text{ in } \Omega, \qquad \mathbf{Q}\cdot\mathbf{n} = 0, \ \ \mathrm{curl}\,\mathbf{Q} = H_{\text{ext}} \ \text{ on } \partial\Omega$$

has a unique solution $\mathbf{Q} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_n^1(\Omega)$ which satisfies the a priori estimate

$$\|\,\mathbf{Q}\,\|_{H^2(\Omega)} \le C\left(\|\,\mathbf{f}\,\|_{L^2(\Omega)} + \|\,H_{\text{ext}}\,\|_{H^1(\Omega)}\right).$$

This property holds, for instance, when the domain Ω has a smooth boundary $C^{2,1}$ [8] or Ω is a rectangular domain [18]. We believe that (H3) holds also for convex polygonal domains. However, we shall not elaborate on this issue here.

To proceed, we set $\mathcal{W}(0,T) = L^2(0,T;\mathcal{H}^1(\Omega)) \cap H^1(0,T;(\mathcal{H}^1(\Omega))')$ and $\mathbf{W}_n(0,T) = L^2(0,T;\mathbf{H}_n^1(\Omega)) \cap H^1(0,T;(\mathbf{H}_n^1(\Omega))')$, where X' stands for the dual space of X. We now give the precise definition of the weak formulation of the problem (1.1)–(1.4) under the gauge choice (1.5).

Continuous problem. Find a pair $(\psi, \mathbf{A}) \in \mathcal{W}(0,T) \times \mathbf{W}_n(0,T)$ such that

$$\psi(\cdot, 0) = \psi_0(\cdot), \quad \mathbf{A}(\cdot, 0) = \mathbf{A}_0(\cdot) \tag{2.2}$$

and

$$\begin{aligned}
&\eta \int_0^T \int_\Omega \partial_t \psi \omega dxdt - \mathbf{i}\eta\kappa \int_0^T \int_\Omega \operatorname{div} \mathbf{A} \psi \omega dxdt \\
&+ \int_0^T \int_\Omega \Big(\frac{\mathbf{i}}{\kappa} \nabla \psi + \mathbf{A}\psi \Big) \Big(-\frac{\mathbf{i}}{\kappa} \nabla \omega + \mathbf{A}\omega \Big) dxdt \\
&+ \int_0^T \int_\Omega (|\psi|^2 - 1)\psi\omega dxdt = 0 \quad \forall \omega \in L^2(0,T;\mathcal{H}^1(\Omega)),
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t \mathbf{A}\mathbf{B} dxdt + \int_0^T \int_\Omega (\operatorname{div} \mathbf{A} \operatorname{div} \mathbf{B} + \operatorname{curl} \mathbf{A} \operatorname{curl} \mathbf{B}) dxdt \\
&+ \int_0^T \int_\Omega \Re\Big[\Big(\frac{\mathbf{i}}{\kappa} \nabla \psi + \mathbf{A}\psi \Big) \bar{\psi} \Big] \mathbf{B} dxdt \\
&= \int_0^T \int_{\partial\Omega} H(\mathbf{B} \cdot \boldsymbol{\tau}) dsdt \quad \forall \mathbf{B} \in L^2(0,T;\mathbf{H}_n^1(\Omega)).
\end{aligned} \tag{2.4}$$

Here $\boldsymbol{\tau} = (n_2, -n_1)$ is the unit tangent to $\partial\Omega$.

Assume from now on that Ω is a convex polygon satisfying (H3). The following lemma can be easily proved by modifying the method in [8].

LEMMA 2.1. *Under the hypotheses* (H1)–(H3)*, the above* continuous problem *has a unique solution* $(\psi, \mathbf{A}) \in \mathcal{W}_2^{2,1}(Q_T) \times \mathbf{W}_2^{2,1}(Q_T)$ *such that* $|\psi| \le 1$ *almost everywhere in* Q_T.

For convenience, we set

$$\begin{aligned}
f(\psi, \mathbf{A}) &= \Big(\frac{\mathbf{i}}{\kappa} - \mathbf{i}\eta\kappa \Big) \operatorname{div} \mathbf{A}\psi + \frac{2\mathbf{i}}{\kappa} \mathbf{A} \cdot \nabla \psi + (|\mathbf{A}|^2 + |\psi|^2 - 1)\psi, \\
\mathbf{g}(\psi, \mathbf{A}) &= \Re\Big[\Big(\frac{\mathbf{i}}{\kappa} \nabla \psi + \mathbf{A}\psi \Big) \bar{\psi} \Big].
\end{aligned}$$

Then (1.1)–(1.4) with gauge choice (1.5) can be written as

$$\eta \partial_t \psi - \frac{1}{\kappa^2} \Delta \psi + f(\psi, \mathbf{A}) = 0 \ \text{ in } Q_T, \tag{2.5}$$

$$\partial_t \mathbf{A} - \Delta \mathbf{A} + \mathbf{g}(\psi, \mathbf{A}) = 0 \ \text{ in } Q_T. \tag{2.6}$$

To conclude this section, we recall the following well-known Nirenberg–Gagliardo inequality:

$$\| u \|_{L^4(\Omega)} \le C \| u \|_{L^2(\Omega)}^{1/2} \| u \|_{H^1(\Omega)}^{1/2} \quad \forall u \in H^1(\Omega). \tag{2.7}$$

3. Discretization. We now introduce the discrete problem, which combines continuous piecewise linear finite elements in space with semi-implicit finite differences in time. We denote by τ_n the nth time-step and set

$$t^n := \sum_{i=1}^{n} \tau_i, \quad u^n(\cdot) := u(\cdot, t^n)$$

for any function u continuous in $(t^{n-1}, t^n]$. Let N be the total number of time-steps, that is, $t^N \geq T$.

We denote by $\mathcal{M}^n$ a uniformly regular partition of Ω into simplexes [10]. The mesh $\mathcal{M}^n$ is obtained by refinement/coarsening of $\mathcal{M}^{n-1}$, and thus $\mathcal{M}^n$ and $\mathcal{M}^{n-1}$ are *compatible*. Given a triangle $S \in \mathcal{M}^n$, h_S stands for its diameter and h_n denotes the mesh density function $h_n|_S = h_S$ for all $S \in \mathcal{M}^n$. We also denote by $\mathcal{B}^n$ the collection of interior boundaries or sides e of $\mathcal{M}^n$ in Ω and $\bar{\mathcal{B}}^n$ the collection of all sides of $\mathcal{M}^n$; h_e stands for the size of $e \in \bar{\mathcal{B}}^n$.

Let V^n indicate the usual space of C^0 piecewise linear finite elements over $\mathcal{M}^n$ and $\mathbf{V}_0^n = \mathbf{V}^n \cap \mathbf{H}_n^1(\Omega)$. Let $I^n : C(\bar{\Omega}) \to V^n$ be the usual Lagrange interpolation operator; then for any $S \in \mathcal{M}^n$ or $e \in \mathcal{B}^n$, the following local approximation properties hold [10]:

$$\| u - I^n u \|_{L^2(S)} \leq C^* h_S^2 \| u \|_{H^2(S)}, \quad \| u - I^n u \|_{L^2(e)} \leq C^* h_e^{3/2} \| u \|_{H^2(\tilde{S})}, \tag{3.1}$$

where $\tilde{S}$ is any element in $\mathcal{M}^n$ with $e \in \mathcal{B}^n$ as part of its boundary. The constant C^* depends only on the minimum angle of the mesh $\mathcal{M}^n$.

Let $\varphi^0 = I^0 \psi_0$ and $\mathbf{D}^0 = I^0 \mathbf{A}_0$. Denote by $\mathcal{P}_n : \mathcal{L}^2(\Omega) \to \mathcal{V}^n$ and $\mathbf{P}_n : \mathbf{L}^2(\Omega) \to \mathbf{V}_0^n$ the L^2-projection operators. The discrete problem approximating (2.2)–(2.4) is defined as follows.

Discrete problem. *Given $(\varphi^{n-1}, \mathbf{D}^{n-1}) \in \mathcal{V}^{n-1} \times \mathbf{V}_0^{n-1}$, then $\mathcal{M}^{n-1}$ and τ_{n-1} are modified as described below to get $\mathcal{M}^n$ and τ_n and thereafter $(\varphi^n, \mathbf{D}^n) \in \mathcal{V}^n \times \mathbf{V}_0^n$ computed according to the following prescription*:

$$\eta \left\langle \frac{\varphi^n - \mathcal{P}_n \varphi^{n-1}}{\tau_n}, \tilde{\omega} \right\rangle + \frac{1}{\kappa^2} \langle \nabla \varphi^n, \nabla \tilde{\omega} \rangle + \langle f(\varphi^n, \mathbf{D}^n), \tilde{\omega} \rangle = 0 \quad \forall \tilde{\omega} \in \mathcal{V}^n, \tag{3.2}$$

$$\begin{aligned} \left\langle \frac{\mathbf{D}^n - \mathbf{P}_n \mathbf{D}^{n-1}}{\tau_n}, \tilde{\mathbf{B}} \right\rangle &+ \langle \operatorname{div} \mathbf{D}^n, \operatorname{div} \tilde{\mathbf{B}} \rangle + \langle \operatorname{curl} \mathbf{D}^n, \operatorname{curl} \tilde{\mathbf{B}} \rangle \\ &+ \langle \mathbf{P}_n[\mathbf{g}(\varphi^{n-1}, \mathbf{D}^{n-1})], \tilde{\mathbf{B}} \rangle = \langle\!\langle I^n H^n, \tilde{\mathbf{B}} \cdot \boldsymbol{\tau} \rangle\!\rangle \quad \forall \tilde{\mathbf{B}} \in \mathbf{V}_0^n. \end{aligned} \tag{3.3}$$

Hereafter, $\langle \cdot, \cdot \rangle$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ stand for the L^2-scalar products in $\mathcal{L}^2(\Omega)$ and $\mathbf{L}^2(\partial\Omega)$, respectively.

We remark that at each time-step n, (3.3) is a linear system of equations with a positive definite coefficient matrix, which can be solved by standard methods. As soon as we know $\mathbf{D}^n$ from (3.3), we substitute it into (3.2) and solve the nonlinear system of equations to obtain φ^n by using, for example, Newton's iterative method [6]. By taking $\tilde{\omega} = \bar{\varphi}^n$ in (3.2) and $\tilde{\mathbf{B}} = \mathbf{D}^n$ in (3.3) we can prove the following uniform estimate for the discrete solution by using the standard energy argument [6]:

$$\max_{1 \leq n \leq N} (\| \varphi^n \|_{L^2(\Omega)} + \| \mathbf{D}^n \|_{L^2(\Omega)}) \leq C, \tag{3.4}$$

where the constant C depends only on η, κ, Ω, T, and the norms of $\psi_0, \mathbf{A}_0, H$ indicated in hypotheses (H1)–(H2).

We now introduce some notation. We define the interior residuals as follows:

$$R_\psi^n := \eta \frac{\varphi^n - \mathcal{P}_n \varphi^{n-1}}{\tau_n} + f(\varphi^n, \mathbf{D}^n), \quad R_{\mathbf{A}}^n := \frac{\mathbf{D}^n - \mathbf{P}_n \mathbf{D}^{n-1}}{\tau_n} + \mathbf{P}_n[\mathbf{g}(\varphi^{n-1}, \mathbf{D}^{n-1})].$$

Let the jump of $\nabla\varphi^n$ across $e \in \mathcal{B}^n$ be

$$\llbracket \nabla\varphi^n \rrbracket_e := (\nabla\varphi^n_{|S_1} - \nabla\varphi^n_{|S_2}) \cdot \mathbf{n}_e. \tag{3.5}$$

Note that with the convention that the unit normal vector $\mathbf{n}_e$ to e points from S_2 to S_1, the jump $\llbracket \nabla\varphi^n \rrbracket_e$ is well defined. Similarly, we define the jumps

$$\llbracket \operatorname{div} \mathbf{D}^n \rrbracket_e := \operatorname{div} \mathbf{D}^n_{|S_1} - \operatorname{div} \mathbf{D}^n_{|S_2}, \qquad \llbracket \operatorname{curl} \mathbf{D}^n \rrbracket_e := \operatorname{curl} \mathbf{D}^n_{|S_1} - \operatorname{curl} \mathbf{D}^n_{|S_2}.$$

Let φ and $\hat\varphi$ denote the piecewise linear and piecewise constant extensions of $\{\varphi^n\}$, that is, $\varphi(\cdot,0) = \hat\varphi(\cdot,0) = \varphi^0(\cdot)$ and, $\forall t^{n-1} < t \le t^n$,

$$\hat\varphi(\cdot,t) := \varphi^n(\cdot) \in \mathcal{V}^n, \quad \varphi(\cdot,t) := \frac{t^n - t}{\tau_n}\varphi^{n-1}(\cdot) + \frac{t - t^{n-1}}{\tau_n}\varphi^n(\cdot).$$

Similarly, we can define $\widehat{\mathbf{D}} \in \mathbf{V}_0^n$ and $\mathbf{D}$. Finally, for any $\gamma > 0$ and $\mathcal{D} \subset \bar\Omega$ we introduce the mesh dependent norms

$$|\!|\!|\, h_n^\gamma \phi \,|\!|\!|_{L^2(\mathcal{D})} := \left(\sum_{e \subset \mathcal{D}, e \in \bar{\mathcal{B}}^n} h_e^{2\gamma} \|\, \phi \,\|^2_{L^2(e)} \right)^{1/2},$$

$$\|\, h_n^\gamma \phi \,\|_{L^2(\mathcal{D})} := \left(\sum_{S \subset \mathcal{D}, S \in \mathcal{M}^n} h_S^{2\gamma} \|\, \phi \,\|^2_{L^2(S)} \right)^{1/2}.$$

4. A dual problem. In this section we introduce and study a dual problem which is the dual of the linearization of (2.2)–(2.4) at $(\psi, \mathbf{A})$. We first formulate the linearization of (2.2)–(2.4) at $(\psi, \mathbf{A})$ as follows: Given $(\psi, \mathbf{A})$ to be the solution of (2.2)–(2.4), find $(\psi^*, \mathbf{A}^*) \in \mathcal{W}(0,T) \times \mathbf{W}_n(0,T)$ such that

$$\psi^*(\cdot,0) = \psi_0^*(\cdot), \quad \mathbf{A}^*(\cdot,0) = \mathbf{A}_0^*(\cdot) \tag{4.1}$$

and

$$\begin{aligned} &\eta\int_0^T\!\!\int_\Omega \partial_t\psi^*\omega dxdt - \mathbf{i}\eta\kappa\int_0^T\!\!\int_\Omega (\operatorname{div}\mathbf{A}^*\psi + \operatorname{div}\mathbf{A}\psi^*)\omega dxdt \\ &+ \int_0^T\!\!\int_\Omega \Big(\frac{\mathbf{i}}{\kappa}\nabla\psi^* + \mathbf{A}^*\psi + \mathbf{A}\psi^*\Big)\Big(-\frac{\mathbf{i}}{\kappa}\nabla\omega + \mathbf{A}\omega\Big)dxdt \\ &+ \int_0^T\!\!\int_\Omega \Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\mathbf{A}^*\omega dxdt \\ &+ \int_0^T\!\!\int_\Omega \Big[(2|\psi|^2 - 1)\psi^* + \psi\bar\psi^*\psi\Big]\omega dxdt = 0 \quad \forall\omega \in L^2(0,T;\mathcal{H}^1(\Omega)), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \partial_t\mathbf{A}\mathbf{B}dxdt + \int_0^T\!\!\int_\Omega (\operatorname{div}\mathbf{A}^*\operatorname{div}\mathbf{B} + \operatorname{curl}\mathbf{A}^*\operatorname{curl}\mathbf{B})dxdt \\ &+ \int_0^T\!\!\int_\Omega \Re\Big[\Big(\frac{\mathbf{i}}{\kappa}\nabla\psi^* + \mathbf{A}^*\psi + \mathbf{A}\psi^*\Big)\bar\psi + \Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\bar\psi^*\Big]\mathbf{B}dtdt \\ &= 0 \qquad \forall\mathbf{B} \in L^2(0,T;\mathbf{H}_n^1(\Omega)). \end{aligned} \tag{4.3}$$

Now we are in the position to define the following dual parabolic problem which is dual to (4.1)–(4.3).

Dual problem. Given $(\theta^*, \mathbf{W}^*) \in \mathcal{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ and an arbitrary $t^* \in (0,T]$, find $(\theta, \mathbf{W}) \in \mathcal{W}(0,T) \times \mathbf{W}_n(0,T)$ such that

$$\theta(\cdot, t^*) = \theta^*(\cdot), \quad \mathbf{W}(\cdot, t^*) = \mathbf{W}^*(\cdot) \tag{4.4}$$

and

$$\begin{aligned}
&-\eta \int_0^{t^*}\int_\Omega \partial_t \theta \omega dx dt - \mathbf{i}\eta\kappa \int_0^{t^*}\int_\Omega \operatorname{div} \mathbf{A}\theta\omega dx dt \\
&+ \int_0^{t^*}\int_\Omega \Big(-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\Big)\Big(\frac{\mathbf{i}}{\kappa}\nabla\omega + \mathbf{A}\omega\Big) dx dt \\
&+ \int_0^{t^*}\int_\Omega \Big[\Big(-\frac{\mathbf{i}}{\kappa}\nabla\bar\psi + \mathbf{A}\bar\psi\Big)\mathbf{W}\omega + \bar\psi\mathbf{W}\Big(\frac{\mathbf{i}}{\kappa}\nabla\omega + \mathbf{A}\omega\Big)\Big] dx dt \\
&+ \int_0^{t^*}\int_\Omega \Big[(2|\psi|^2 - 1)\theta + \bar\psi\bar\theta\bar\psi\Big]\omega dx dt = 0 \quad \forall \omega \in L^2(0,t^*;\mathcal{H}^1(\Omega)),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&-\int_0^{t^*}\int_\Omega \partial_t \mathbf{W}\mathbf{B} dx dt + \int_0^{t^*}\int_\Omega (\operatorname{div}\mathbf{W}\operatorname{div}\mathbf{B} + \operatorname{curl}\mathbf{W}\operatorname{curl}\mathbf{B}) dx dt \\
&+ \int_0^{t^*}\int_\Omega \Re\Big[\mathbf{i}\eta\kappa\nabla(\psi\theta)\Big]\mathbf{B} dx dt + \int_0^{t^*}\int_\Omega |\psi|^2 \mathbf{W}\mathbf{B} dx dt \\
&+ \int_0^{t^*}\int_\Omega \Re\Big[\Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\theta + \Big(-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\Big)\psi\Big]\mathbf{B} dx dt \\
&= 0 \quad \forall \mathbf{B} \in L^2(0,t^*;\mathbf{H}_n^1(\Omega)).
\end{aligned} \tag{4.6}$$

The linearization problem (4.1)–(4.3) and its dual problem (4.4)–(4.6) have been studied in [7] in the context of exploring the possibility of controlling the motion of vortices in the superconductors through the external magnetic field. The purpose in this section is to derive strong stability estimates for (4.4)–(4.6) which will be used in the next section in the a posteriori error analysis. Throughout we denote by C the generic constant which may depend on η, κ, Ω, T, and the norms of $\psi_0, \mathbf{A}_0, H$ indicated in hypotheses (H1)–(H2).

We will extend the method in [21] for linear heat equation to derive strong stability estimates for $(\theta, \mathbf{W})$ under weak regularity assumption $(\theta^*, \mathbf{W}^*) \in \mathcal{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$. First we note that by Lemma 2.1 we have

$$\|\psi\|_{W_2^{2,1}(Q_T)} + \|\mathbf{A}\|_{W_2^{2,1}(Q_T)} \le C, \quad \|\nabla\psi\|_{L^4(Q_T)} + \|\nabla\mathbf{A}\|_{L^4(Q_T)} \le C. \tag{4.7}$$

The latter estimate in (4.7) follows from the former one and the Nirenberg–Gagliardo inequality (2.7).

LEMMA 4.1. *The following stability estimates are valid* $\forall 0 \le t^* \le T$*:*

$$\begin{aligned}
&\max_{0\le t\le t^*}\Big(\|\theta\|_{L^2(\Omega)}^2 + \|\mathbf{W}\|_{L^2(\Omega)}^2\Big) + \int_0^{t^*}\Big(\|\nabla\theta\|_{L^2(\Omega)}^2 + \|\mathbf{W}\|_{H^1(\Omega)}^2\Big) dt \\
&\le C\Big(\|\theta^*\|_{L^2(\Omega)}^2 + \|\mathbf{W}^*\|_{L^2(\Omega)}^2\Big).
\end{aligned}$$

Proof. Denote by $\chi_{(t,t^*]}$ the characteristic function of the interval $(t,t^*]$. We let $\omega = \bar\theta\chi_{(t,t^*]}$ in (4.5) and take the real part of the obtained equation to obtain

$$\frac{\eta}{2}\|\theta\|_{L^2(\Omega)}^2 - \frac{\eta}{2}\|\theta^*\|_{L^2(\Omega)}^2 + \int_t^{t^*}\left\|-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\right\|_{L^2(\Omega)}^2 dt \tag{4.8}$$

$$= -\int_t^{t^*}\int_\Omega \Re\Big[\Big(-\frac{\mathbf{i}}{\kappa}\nabla\bar\psi + \mathbf{A}\bar\psi\Big)\mathbf{W}\bar\theta + \bar\psi\mathbf{W}\Big(\frac{\mathbf{i}}{\kappa}\nabla\bar\theta + \mathbf{A}\bar\theta\Big)\Big]dxdt$$
$$-\int_t^{t^*}\int_\Omega \Re\Big[(2|\psi|^2-1)|\theta|^2 + \bar\psi^2\bar\theta^2\Big]dxdt.$$

By using (2.1) and (2.7) we can bound the first term on the right-hand side as follows:

$$-\int_t^{t^*}\int_\Omega \Re\Big[\Big(-\frac{\mathbf{i}}{\kappa}\nabla\bar\psi + \mathbf{A}\bar\psi\Big)\mathbf{W}\bar\theta\Big]dxdt$$
$$\le C\int_t^{t^*} \Big\|\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big\|_{L^4(\Omega)} \|\mathbf{W}\|_{L^2(\Omega)} \|\theta\|_{L^2(\Omega)}\,dt$$
$$+ C\int_t^{t^*} \Big\|\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big\|_{L^4(\Omega)} \|\mathbf{W}\|_{L^2(\Omega)}^{1/2}\Big(\|\operatorname{div}\mathbf{W}\|_{L^2(\Omega)}^{1/2} + \|\operatorname{curl}\mathbf{W}\|_{L^2(\Omega)}^{1/2}\Big)\|\theta\|_{L^2(\Omega)}$$
$$\le \delta\int_t^{t^*}\Big(\|\operatorname{div}\mathbf{W}\|_{L^2(\Omega)}^2 + \|\operatorname{curl}\mathbf{W}\|_{L^2(\Omega)}^2\Big)dt + \frac{C}{\delta}\int_t^{t^*}\|\mathbf{W}\|_{L^2(\Omega)}^2\,dt$$
$$+ C\int_t^{t^*}\Big\|\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big\|_{L^4(\Omega)}^2\|\theta\|_{L^2(\Omega)}^2\,dt \quad \forall\delta > 0.$$

By using the fact that $|\psi| \le 1$ a.e. in Q_T we easily get

$$-\int_t^{t^*}\int_\Omega \Re\Big[\bar\psi\mathbf{W}\Big(\frac{\mathbf{i}}{\kappa}\nabla\bar\theta + \mathbf{A}\bar\theta\Big)\Big]dxdt \le \frac12\int_t^{t^*}\Big\|-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\Big\|_{L^2(\Omega)}^2 dt$$
$$+ C\int_0^{t^*}\|\mathbf{W}\|_{L^2(\Omega)}^2\,dt$$

and

$$-\int_t^{t^*}\int_\Omega \Re\Big[(2|\psi|^2-1)|\theta|^2 + \bar\psi^2\bar\theta^2\Big]dxdt \le C\int_t^{t^*}\|\theta\|_{L^2(\Omega)}^2\,dt.$$

Inserting these estimates into (4.8) we get

$$\frac{\eta}{2}\|\theta\|_{L^2(\Omega)}^2 - \frac{\eta}{2}\|\theta^*\|_{L^2(\Omega)}^2 + \frac12\int_t^{t^*}\Big\|-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\Big\|_{L^2(\Omega)}^2 dt \tag{4.9}$$
$$\le \delta\int_t^{t^*}\Big(\|\operatorname{div}\mathbf{W}\|_{L^2(\Omega)}^2 + \|\operatorname{curl}\mathbf{W}\|_{L^2(\Omega)}^2\Big)dt + \frac{C}{\delta}\int_t^{t^*}\|\mathbf{W}\|_{L^2(\Omega)}^2\,dt$$
$$+ C\int_t^{t^*}\Big(1 + \Big\|\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big\|_{L^4(\Omega)}^2\Big)\|\theta\|_{L^2(\Omega)}^2\,dt.$$

Similarly, by letting $\mathbf{B} = \mathbf{W}\chi_{(t,t^*]}$ in (4.6) we can obtain that

$$\frac12\|\mathbf{W}\|_{L^2(\Omega)}^2 - \frac12\|\mathbf{W}^*\|_{L^2(\Omega)}^2 + \frac12\int_t^{t^*}\Big(\|\operatorname{div}\mathbf{W}\|_{L^2(\Omega)}^2 + \|\operatorname{curl}\mathbf{W}\|_{L^2(\Omega)}^2\Big)dt \tag{4.10}$$
$$\le \delta\int_t^{t^*}\Big\|-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta\Big\|_{L^2(\Omega)}^2 dt + \frac{C}{\delta}\int_t^{t^*}\|\mathbf{W}\|_{L^2(\Omega)}^2\,dt$$
$$+ C\int_t^{t^*}\Big(1 + \Big\|\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\Big\|_{L^4(\Omega)}^2\Big)\|\theta\|_{L^2(\Omega)}^2\,dt.$$

Now observe that by (4.7)

$$\int_0^T \left\| \frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi \right\|_{L^4(\Omega)}^2 dt \le C.$$

We conclude after adding (4.9) and (4.10), taking δ appropriately small, and using Gronwall inequality that

$$\begin{aligned}
&\max_{0\le t\le t^*} \|\,\theta\,\|_{L^2(\Omega)}^2 + \int_0^{t^*} \left\| -\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta \right\|_{L^2(\Omega)}^2 dt \\
&+ \max_{0\le t\le t^*} \|\,\mathbf{W}\,\|_{L^2(\Omega)}^2 + \int_0^{t^*} \left(\|\operatorname{div}\mathbf{W}\,\|_{L^2(\Omega)}^2 + \|\operatorname{curl}\mathbf{W}\,\|_{L^2(\Omega)}^2 \right) dt \\
&\le C\left(\|\,\theta^*\,\|_{L^2(\Omega)}^2 + \|\,\mathbf{W}^*\,\|_{L^2(\Omega)}^2 \right).
\end{aligned}$$

Now the lemma follows by observing that

$$\begin{aligned}
\int_0^{t^*} \|\nabla\theta\,\|_{L^2(\Omega)}^2 dt &\le C\int_0^{t^*} \left\| -\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta \right\|_{L^2(\Omega)}^2 dt + C\int_0^{t^*} \|\,\mathbf{A}\,\|_{L^\infty(\Omega)}^2 \|\,\theta\,\|_{L^2(\Omega)}^2 dt \\
&\le C\int_0^{t^*} \left\| -\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta \right\|_{L^2(\Omega)}^2 dt + C\max_{0\le t\le t^*} \|\,\theta\,\|_{L^2(\Omega)}^2 dt.
\end{aligned}$$

This completes the proof. ∎

COROLLARY 4.1. *The following stability estimates are valid* $\forall 0 < t^* \le T$:

$$\int_0^{t^*} \left(\|\,\theta\,\|_{L^4(\Omega)}^4 + \|\,\mathbf{W}\,\|_{L^4(\Omega)}^4 \right) dt \le C\left(\|\,\theta^*\,\|_{L^2(\Omega)}^2 + \|\,\mathbf{W}^*\,\|_{L^2(\Omega)}^2 \right)^2.$$

Proof. This is a direct consequence of Lemma 4.1 and (2.7). ∎

To proceed, we set

$$\begin{aligned}
f_1(\psi,\mathbf{A};\theta,\mathbf{W}) = -\mathbf{i}\eta\kappa\operatorname{div}\mathbf{A}\theta - \frac{\mathbf{i}}{\kappa}\operatorname{div}(\mathbf{A}\theta) + \mathbf{A}\Big(-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta \Big) \\
+ \Big(-\frac{\mathbf{i}}{\kappa}\nabla\bar\psi + \mathbf{A}\bar\psi \Big)\mathbf{W} + \Big(-\frac{\mathbf{i}}{\kappa}\nabla + \mathbf{A} \Big)(\bar\psi\mathbf{W}) + (2|\psi|^2 - 1)\theta + \bar\psi^2\bar\theta
\end{aligned}$$

and

$$\mathbf{g}_1(\psi,\mathbf{A};\theta,\mathbf{W}) = \Re\Big[\mathbf{i}\eta\kappa\nabla(\psi\theta) + \Big(\frac{\mathbf{i}}{\kappa}\nabla\psi + \mathbf{A}\psi \Big)\theta + \Big(-\frac{\mathbf{i}}{\kappa}\nabla\theta + \mathbf{A}\theta \Big)\psi \Big] + |\psi|^2\mathbf{W}.$$

Then it is easy to see that (4.4)–(4.6) is the weak formulation of the following linear parabolic equations:

$$-\eta\partial_t\theta - \frac{1}{\kappa^2}\Delta\theta + f_1(\psi,\mathbf{A};\theta,\mathbf{W}) = 0 \quad \text{in } Q^*, \tag{4.11}$$

$$-\partial_t\mathbf{W} - \Delta\mathbf{W} + \mathbf{g}_1(\psi,\mathbf{A};\theta,\mathbf{W}) = 0 \quad \text{in } Q^*, \tag{4.12}$$

$$\nabla\theta\cdot\mathbf{n} = 0, \quad \mathbf{W}\cdot\mathbf{n} = 0, \quad \operatorname{curl}\mathbf{W} = 0 \quad \text{on } \Gamma^*, \tag{4.13}$$

$$\theta(\cdot,t^*) = \theta^*(\cdot), \quad \mathbf{W}(\cdot,t^*) = \mathbf{W}^*(\cdot) \quad \text{on } \Omega, \tag{4.14}$$

where $Q^* = \Omega\times(0,t^*)$ and $\Gamma^* = \partial\Omega\times(0,t^*)$.

LEMMA 4.2. *The following stability estimates are valid* $\forall 0 < t^* \le T$*:*

$$\max_{0\le t\le t^*}(t^*-t)\Big(\|\nabla\theta\|^2_{L^2(\Omega)}+\|\operatorname{div}\mathbf{W}\|^2_{L^2(\Omega)}+\|\operatorname{curl}\mathbf{W}\|^2_{L^2(\Omega)}\Big)$$
$$+\int_0^{t^*}(t^*-t)\Big(\|\partial_t\theta\|^2_{L^2(\Omega)}+\|\partial_t\mathbf{W}\|^2_{L^2(\Omega)}\Big)dt$$
$$\le C\left(\|\theta^*\|^2_{L^2(\Omega)}+\|\mathbf{W}^*\|^2_{L^2(\Omega)}\right).$$

Proof. First we know from Lemma 4.1 that there exists a sequence $t_j^* \nearrow t^*$ such that

$$(4.15)\quad (t^*-t_j^*)\Big(\big\|\nabla\theta(t_j^*)\big\|^2_{L^2(\Omega)}+\big\|\operatorname{div}\mathbf{W}(t_j^*)\big\|^2_{L^2(\Omega)}+\big\|\operatorname{curl}\mathbf{W}(t_j^*)\big\|^2_{L^2(\Omega)}\Big)\to 0.$$

We multiply (4.11) by $-(t^*-t)\partial_t\bar\theta$, integrate over $\Omega\times(s,t_j^*]$, and then take the real part of the equation to obtain that

$$(4.16)\quad \eta\int_s^{t_j^*}(t^*-t)\|\partial_t\theta\|^2_{L^2(\Omega)}dt=\frac{1}{2\kappa^2}\int_s^{t_j^*}(t^*-t)\frac{d}{dt}\|\nabla\theta\|^2_{L^2(\Omega)}dt$$
$$+\int_s^{t_j^*}\int_\Omega(t^*-t)\Re\Big[f_1(\psi,\mathbf{A};\theta,\mathbf{W})\partial_t\bar\theta\Big]dt.$$

Integrating by parts and using Lemma 4.1 we have

$$(4.17)\quad \int_s^{t_j^*}(t^*-t)\frac{d}{dt}\|\nabla\theta\|^2_{L^2(\Omega)}dt$$
$$=\int_s^{t_j^*}\|\nabla\theta\|^2_{L^2(\Omega)}dt+(t^*-t_j^*)\big\|\nabla\theta(t_j^*)\big\|^2_{L^2(\Omega)}-(t^*-s)\|\nabla\theta(s)\|^2_{L^2(\Omega)}$$
$$\le(t^*-t_j^*)\big\|\nabla\theta(t_j^*)\big\|^2_{L^2(\Omega)}-(t^*-s)\|\nabla\theta(s)\|^2_{L^2(\Omega)}$$
$$+C(\|\theta^*\|^2_{L^2(\Omega)}+\|\mathbf{W}^*\|^2_{L^2(\Omega)}).$$

Next, by Lemma 4.1, we easily get that

$$(4.18)\quad \int_s^{t_j^*}\int_\Omega(t^*-t)\Re\big[f_1(\psi,\mathbf{A};\theta,\mathbf{W})\partial_t\bar\theta\Big]dt$$
$$\le\frac{\eta}{2}\int_s^{t_j^*}(t^*-t)\|\partial_t\theta\|^2_{L^2(\Omega)}dt+C\int_0^{t^*}(t^*-t)\|f_1(\psi,\mathbf{A};\theta,\mathbf{W})\|^2_{L^2(\Omega)}dt$$
$$\le\frac{\eta}{2}\int_s^{t_j^*}(t^*-t)\|\partial_t\theta\|^2_{L^2(\Omega)}dt+C\left(\|\theta^*\|^2_{L^2(\Omega)}+\|\mathbf{W}^*\|^2_{L^2(\Omega)}\right).$$

For example, we can estimate the first term in $f_1(\psi,\mathbf{A};\theta,\mathbf{W})$ by using (4.7) and Corollary 4.1 as follows:

$$\int_0^{t^*}\|\mathbf{i}\eta\kappa\operatorname{div}\mathbf{A}\theta\|^2_{L^2(\Omega)}dt\le C\left(\int_0^T\|\operatorname{div}\mathbf{A}\|^4_{L^4(\Omega)}dt\right)^{1/2}\left(\int_0^{t^*}\|\theta\|^4_{L^4(\Omega)}dt\right)^{1/2}$$
$$\le C\left(\|\theta^*\|^2_{L^2(\Omega)}+\|\mathbf{W}^*\|^2_{L^2(\Omega)}\right).$$

Substituting (4.17)–(4.18) into (4.16), letting $t_j^* \nearrow t^*$, and using (4.15), we get

$$\frac{\eta}{2}\int_s^{t^*} \| \partial_t \theta \|^2_{L^2(\Omega)}\, dt + (t^* - s)\, \| \nabla\theta(s) \|^2_{L^2(\Omega)} \le C\left(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\right).$$

Similarly, we obtain from (4.12) that

$$\frac{1}{2}\int_s^{t^*} (t^* - t)\, \| \partial_t \mathbf{W} \|^2_{L^2(\Omega)}\, dt + (t^* - s)\left(\| \operatorname{div} \mathbf{W} \|^2_{L^2(\Omega)} + \| \operatorname{curl} \mathbf{W} \|^2_{L^2(\Omega)}\right)$$
$$\le C\left(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\right).$$

This completes the proof of Lemma 4.2. □

LEMMA 4.3. *The following stability estimates are valid* $\forall 0 < t^* \le T$*:*

$$\int_0^{t^*} (t^* - t)\Big(\| \theta \|^2_{H^2(\Omega)} + \| \mathbf{W} \|^2_{H^2(\Omega)}\Big)dt \le C\Big(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\Big).$$

Proof. This is a direct consequence of Lemma 4.2. For instance, since

$$\int_0^{t^*} \| \mathbf{g}_1(\psi, \mathbf{A}; \theta, \mathbf{W}) \|^2_{L^2(\Omega)}\, dt \le C\left(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\right),$$

we know from (4.12)–(4.13) and Lemma 4.2 that

$$\int_0^{t^*} (t^* - t)\, \| \Delta \mathbf{W} \|^2_{L^2(\Omega)} \le C\left(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\right).$$

This, by (H3), yields

$$\int_0^{t^*} (t^* - t)\| \mathbf{W} \|^2_{H^2(\Omega)} dt \le C\left(\| \theta^* \|^2_{L^2(\Omega)} + \| \mathbf{W}^* \|^2_{L^2(\Omega)}\right).$$

This completes the proof. □

The following corollary follows directly from Lemmas 4.2 and 4.3.

COROLLARY 4.2. *The following stability estimates are valid* $\forall t^{m-1} < t^* \le T$*:*

$$\int_0^{t^{m-1}} \Big(\| \partial_t \theta \|_{L^2(\Omega)} + \| \partial_t \mathbf{W} \|_{L^2(\Omega)} + \| \theta \|_{H^2(\Omega)} + \| \mathbf{W} \|_{H^2(\Omega)}\Big)dt$$
$$\le C\Big(\log\frac{t^*}{t^* - t^{m-1}}\Big)^{1/2}\left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)}\right).$$

Now, for any $t^* \in (t^{m-1}, t^m]$, we denote by $t^n \wedge t^* = \min(t^n, t^*)$ and

$$\theta^n_{\rm int} = \int_{t^{n-1}}^{t^n \wedge t^*} \theta dt, \quad \mathbf{W}^n_{\rm int} = \int_{t^{n-1}}^{t^n \wedge t^*} \mathbf{W} dt.$$

LEMMA 4.4. *The following estimates are valid for any* $t^* \in (t^{m-1}, t^m]$*:*

$$\sum_{n=1}^m \left(\| \theta^n_{\rm int} \|_{H^2(\Omega)} + \| \mathbf{W}^n_{\rm int} \|_{H^2(\Omega)}\right)$$
$$\le C\Big(1 + \log\frac{t^*}{t^* - t^{m-1}}\Big)^{1/2}\left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)}\right).$$

Proof. We prove only the estimate for $\theta^n_{\rm int}$ since the estimate for $\mathbf{W}^n_{\rm int}$ is similar. First from Corollary 4.2, Lemma 4.1, and elliptic regularity theory [18] we have that

$$
\begin{aligned}
(4.19)\qquad & \sum_{n=1}^{m} \| \theta^n_{\rm int} \|_{H^2(\Omega)} \\
&\le \int_0^{t^{m-1}} \| \theta \|_{H^2(\Omega)} dt + \left\| \int_{t^{m-1}}^{t^*} \theta dt \right\|_{H^2(\Omega)} \\
&\le \int_0^{t^{m-1}} \| \theta \|_{H^2(\Omega)} dt + C \left\| \int_{t^{m-1}}^{t^*} \theta dt \right\|_{L^2(\Omega)} + C \left\| \int_{t^{m-1}}^{t^*} \Delta\theta dt \right\|_{L^2(\Omega)} \\
&\le C \left(1 + \log \frac{t^*}{t^* - t^{m-1}} \right)^{1/2} \left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)} \right) \\
&\quad + C \left\| \int_{t^{m-1}}^{t^*} \Delta\theta dt \right\|_{L^2(\Omega)} .
\end{aligned}
$$

However, from (4.11) and Lemma 4.1 we have

$$
\begin{aligned}
(4.20)\qquad & \left\| \int_{t^{m-1}}^{t^*} \Delta\theta dt \right\|_{L^2(\Omega)} \\
&\le C \left\| \int_{t^{m-1}}^{t^*} \partial_t\theta dt \right\|_{L^2(\Omega)} + C \left\| \int_{t^{m-1}}^{t^*} f_1(\psi, \mathbf{A}; \theta, \mathbf{W}) dt \right\|_{L^2(\Omega)} \\
&\le C \left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)} \right) + C \int_{t^{m-1}}^{t^*} \| f_1(\psi, \mathbf{A}; \theta, \mathbf{W}) \|_{L^2(\Omega)} \, dt \\
&\le C \left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)} \right).
\end{aligned}
$$

For example, the first term in $f_1(\psi, \mathbf{A}; \theta, \mathbf{W})$ can be estimated by using (4.7) and Corollary 4.1 as follows:

$$
\begin{aligned}
\int_{t^{m-1}}^{t^*} \| \mathbf{i}\eta\kappa {\rm div}\, \mathbf{A}\theta \|_{L^2(\Omega)} \, dt &\le C \| \,{\rm div}\, \mathbf{A} \|_{L^4(Q_T)} \| \theta \|_{L^4(Q^*)} \\
&\le C \left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)} \right).
\end{aligned}
$$

Inserting (4.20) into (4.19) we finally obtain that

$$
\sum_{n=1}^{m} \| \theta^n_{\rm int} \|_{H^2(\Omega)} \le C \left(1 + \log \frac{t^*}{t^* - t^{m-1}} \right)^{1/2} \left(\| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)} \right).
$$

This completes the proof. ∎

5. A posteriori error analysis. We first denote by the error functions

$$
e_\psi := \psi - \varphi, \quad \hat{e}_\psi = \psi - \hat{\varphi}; \quad e_{\mathbf{A}} := \mathbf{A} - \mathbf{D}, \quad \hat{e}_{\mathbf{A}} := \mathbf{A} - \widehat{\mathbf{D}}.
$$

Let $h_{\max} = \max_{1\le n\le N} \bar{h}_n$, where $\bar{h}_n = \max_{S\in\mathcal{M}^n} h_S$, and $\tau_{\max} = \max_{1\le n\le N} \tau_n$. We need the following assumption on the discrete solutions.

(H4) The quantity $\Lambda = \max_{0\le t^*\le T}\Lambda(t^*)$, where

$$\Lambda(t^*) = \left(\int_0^{t^*}\left(\|\,\hat e_\psi\,\|^2_{H^1(\Omega)} + \|\,\hat e_{\mathbf A}\,\|^2_{H^1(\Omega)}\right)dt\right)^3 \Big/ \max_{0\le t\le t^*}\left(\|\,\hat e_\psi\,\|^2_{L^2(\Omega)} + \|\,\hat e_{\mathbf A}\,\|^2_{L^2(\Omega)}\right)$$

tends to zero as $(h_{\max}, \tau_{\max}) \to 0$.

We remark that this assumption is in fact a *nondegeneracy* assumption. By the a priori error analysis in [6], the following optimal energy a priori error estimates hold under suitable restriction on the mesh changes:

$$\begin{aligned}\max_{0\le t\le t^m}&\left(\|\,\hat e_\psi\,\|^2_{L^2(\Omega)} + \|\,\hat e_{\mathbf A}\,\|^2_{L^2(\Omega)}\right)\\ &+\int_0^{t^m}\left(\|\,\hat e_\psi\,\|^2_{H^1(\Omega)} + \|\,\hat e_{\mathbf A}\,\|^2_{H^1(\Omega)}\right)dt \le C\max_{1\le n\le m}(\bar h_n^2 + \tau_n^2).\end{aligned} \tag{5.1}$$

Now (H4) is a direct consequence of the following nondegeneracy assumption: There exists a constant C_* independent of $1 \le m \le N$ such that, for any $t^* \in (t^{m-1}, t^m]$,

$$\max_{0\le t\le t^*}\left(\|\,\hat e_\psi\,\|_{L^2(\Omega)} + \|\,\hat e_{\mathbf A}\,\|_{L^2(\Omega)}\right) \ge C_*\max_{1\le n\le m}(\bar h_n^2 + \tau_n). \tag{5.2}$$

We note that $\max_{1\le n\le m}(\bar h_n^2 + \tau_n)$ is the expected optimal $L^\infty L^2$ error estimate for $\hat e_\psi$ and $\hat e_{\mathbf A}$. Moreover, following from (5.1)–(5.2) we now have

$$\Lambda(t^*) \le C\max_{1\le n\le m}(\bar h_n^2 + \tau_n^2) \qquad \forall t^* \in (t^{m-1}, t^m],$$

which clearly tends to zero as $(h_{\max}, \tau_{\max}) \to 0$.

Let $\bar h_k = \max_{1\le n\le m}\bar h_n$. The inequality (5.2) is guaranteed [1], for example, if $|D_{x_ix_j}\psi(x,t^k)| + |D_{x_ix_j}\mathbf A(x,t^k)| \ge C > 0$ $\forall x$ in a region $D_1 \subset \Omega$, where the local mesh size is larger than $C\bar h_k$, and $|D_t\psi(x,t)| + |D_t\mathbf A(x,t)| \ge C > 0$ for all (x,t) in some cylinder domain $D_2 \times [t_1,t_2] \subset \Omega \times [0,t^*]$, where the local time-step size is bounded below by $C\max_{1\le n\le m}\tau_n$. Therefore, given the vortex structure of the solutions of the time-dependent Ginzburg–Landau model, the assumption (5.2) is not very restrictive. In the extreme case when the solution ψ and $\mathbf A$ are both linear in space and constant in time, the assumption (5.2) is not valid as we might have $\hat e_\psi = 0$ and $\hat e_{\mathbf A} = 0$. However, in this case, the higher order terms $\mathcal R_7(\theta, \mathbf W)$ and $\mathcal R_8(\theta, \mathbf W)$ in the error representation formula below vanish and thus assumption (H4) is no longer necessary in deriving the a posteriori error estimate.

THEOREM 5.1. *Let* (H1)–(H4) *be satisfied. Then there exist two positive constants* h^*, τ^* *and a constant* C *depending only on* $h^*, \tau^*, \eta, \kappa, \Omega, T$, *the norms of* $\psi_0, \mathbf A_0, H$ *indicated in hypotheses* (H1)–(H2), *and the minimum angle of the mesh* $\mathcal M^n$ *such that for* $h_{\max} \le h^*$ *and* $\tau_{\max} \le \tau^*$ *the following a posteriori error estimates hold for any* $0 \le t^m \le T$:

$$\|\,\psi^m - \varphi^m\,\|_{L^2(\Omega)} + \|\,\mathbf A^m - \mathbf D^m\,\|_{L^2(\Omega)} \le C\mathcal E_0 + C\Big(1 + \log\frac{t^m}{\tau_m}\Big)^{1/2}\sum_{i=1}^6\mathcal E_i,$$

where $\mathcal{E}_i = \max_{1\le n\le m} \mathcal{E}_i^n$ *and the error indicators* $\mathcal{E}_0$ *and* $\mathcal{E}_i^n$ *are given by*

$$\mathcal{E}_0 := \| \psi_0 - I^0\psi_0 \|_{L^2(\Omega)} + \| \mathbf{A}_0 - I^0\mathbf{A}_0 \|_{L^2(\Omega)} \qquad \textit{initial error,}$$

$$\mathcal{E}_1^n := \| \varphi^n - \mathcal{P}_n\varphi^{n-1} \|_{L^2(\Omega)} + \| \mathbf{D}^n - \mathbf{P}_n\mathbf{D}^{n-1} \|_{L^2(\Omega)}$$
$$+ \ \| \mathbf{g}(\varphi^n, \mathbf{D}^n) - \mathbf{P}_n[\mathbf{g}(\varphi^{n-1}, \mathbf{D}^{n-1})] \|_{L^2(\Omega)} \qquad \textit{time residual,}$$

$$\mathcal{E}_2^n := + \| \mathbf{D}^{n-1} - \mathbf{P}_n\mathbf{D}^{n-1} \|_{L^2(\Omega)} + \| \varphi^{n-1} - \mathcal{P}_n\varphi^{n-1} \|_{L^2(\Omega)}$$
$$+ \left\| \frac{h_n^2}{\tau_n}\Big(\mathbf{D}^n - \mathbf{P}_n\mathbf{D}^{n-1}\Big) \right\|_{L^2(\Omega)}$$
$$+ \left\| \frac{h_n^2}{\tau_n}\Big(\varphi^{n-1} - \mathcal{P}_n\varphi^{n-1}\Big) \right\|_{L^2(\Omega)} \qquad \textit{coarsening,}$$

$$\mathcal{E}_3^n := \| h_n^2 R_\psi^n \|_{L^2(\Omega)} + \| h_n^2 R_{\mathbf{A}}^n \|_{L^2(\Omega)} \qquad \textit{interior residual,}$$

$$\mathcal{E}_4^n := ||| h_n^{3/2} [\![\nabla\varphi^n]\!]_e |||_{L^2(\Omega)} + ||| h_n^{3/2} [\![\operatorname{div}\mathbf{D}^n]\!]_e |||_{L^2(\Omega)}$$
$$+ ||| h_n^{3/2} [\![\operatorname{curl}\mathbf{D}^n]\!]_e |||_{L^2(\Omega)} \qquad \textit{jump residual,}$$

$$\mathcal{E}_5^n := ||| h_n^{3/2} \nabla\varphi^n \cdot \mathbf{n} |||_{L^2(\partial\Omega)}$$
$$+ ||| h_n^{3/2} (\operatorname{curl}\mathbf{D}^n - I^n H^n) |||_{L^2(\partial\Omega)} \qquad \textit{boundary residual,}$$

$$\mathcal{E}_6^n := \max_{t^{n-1}\le t\le t^n} \| H - I^n H^n \|_{L^2(\partial\Omega)} \qquad \textit{boundary error.}$$

We first derive an error representation formula using the dual problem in section 3 and then use the Galerkin orthogonality property to complete the proof.

5.1. Error representation formula. To prove Theorem 5.1 we first derive an explicit representation formula for the error $\| \psi - \varphi \|_{L^2(\Omega)}$ and $\| \mathbf{A} - \mathbf{D} \|_{L^2(\Omega)}$ based on the linear dual problem (4.4)–(4.6). We first multiply (4.11) by $(\psi - \hat{\varphi})$, (4.12) by $(\mathbf{A} - \widehat{\mathbf{D}})$, and integrate in space and in time from 0 to t^*. We examine the various contributions in turn. Since $\hat{\varphi}, \widehat{\mathbf{D}}$ are piecewise constant in time, we have

$$-\eta \int_0^{t^*} \langle \partial_t\theta, \psi - \hat{\varphi}\rangle dt = \eta \int_0^{t^*} \Big(\langle \theta, \partial_t(\psi - \varphi)\rangle + \langle \partial_t\theta, \hat{\varphi} - \varphi\rangle\Big) dt$$
$$+ \eta\langle \theta^0, \psi_0 - \varphi^0\rangle - \eta\langle \theta^*, \psi(t^*) - \varphi(t^*)\rangle$$

and

$$-\int_0^{t^*} \Big\langle \partial_t\mathbf{W}, \mathbf{A} - \widehat{\mathbf{D}}\Big\rangle dt = \int_0^{t^*} \Big(\langle \mathbf{W}, \partial_t(\mathbf{A} - \mathbf{D})\rangle + \Big\langle \partial_t\mathbf{W}, \widehat{\mathbf{D}} - \mathbf{D}\Big\rangle \Big) dt$$
$$+ \langle \mathbf{W}^0, \mathbf{A}_0 - \mathbf{D}^0\rangle - \langle \mathbf{W}^*, \mathbf{A}(t^*) - \mathbf{D}(t^*)\rangle.$$

Integrating by parts we get

$$-\int_0^{t^*} \frac{1}{\kappa^2}\langle \Delta\theta, \psi - \hat{\varphi}\rangle dt = \int_0^{t^*} \frac{1}{\kappa^2}\langle \nabla\theta, \nabla(\psi - \hat{\varphi})\rangle dt,$$
$$-\int_0^{t^*} \left\langle \Delta \mathbf{W}, \mathbf{A} - \widehat{\mathbf{D}}\right\rangle dt = \int_0^{t^*} \left(\left\langle \operatorname{div}\mathbf{W}, \operatorname{div}\left(\mathbf{A} - \widehat{\mathbf{D}}\right)\right\rangle + \left\langle \operatorname{curl}\mathbf{W}, \operatorname{curl}\left(\mathbf{A} - \widehat{\mathbf{D}}\right)\right\rangle\right) dt.$$

Collecting these equalities and making use of (2.5)–(2.6), we easily end up with

$$\begin{aligned}(5.3)\qquad & \eta \,\|\, \psi(t^*) - \varphi(t^*) \,\|_{L^2(\Omega)} + \|\, \mathbf{A}(t^*) - \mathbf{D}(t^*) \,\|_{L^2(\Omega)} \\ &\le 2 \sup_{(\theta^*,\mathbf{W}^*)\in\mathcal{L}^2(\Omega)\times\mathbf{L}^2(\Omega)} \frac{\Re\Big[\eta\langle\theta^*, \psi(t^*) - \varphi(t^*)\rangle + \langle \mathbf{W}^*, \mathbf{A}(t^*) - \mathbf{D}(t^*)\rangle\Big]}{\|\,\theta^*\,\|_{L^2(\Omega)} + \|\,\mathbf{W}^*\,\|_{L^2(\Omega)}} \\ &= 2 \sup_{(\theta^*,\mathbf{W}^*)\in\mathcal{L}^2(\Omega)\times\mathbf{L}^2(\Omega)} \frac{\Re[\mathcal{R}(\theta, \mathbf{W})]}{\|\,\theta^*\,\|_{L^2(\Omega)} + \|\,\mathbf{W}^*\,\|_{L^2(\Omega)}},\end{aligned}$$

where $\mathcal{R}$, the parabolic residual, is the following distribution:

$$\begin{aligned}(5.4)\ \ & \mathcal{R}(\theta, \mathbf{W}) \\ &= \eta\langle\theta^0, \psi_0 - \varphi^0\rangle + \langle \mathbf{W}^0, \mathbf{A}_0 - \mathbf{D}^0\rangle \\ &+ \int_0^{t^*} \left(\eta\,\langle\partial_t\theta, \hat{\varphi} - \varphi\rangle + \left\langle \partial_t\mathbf{W}, \widehat{\mathbf{D}} - \mathbf{D}\right\rangle\right) dt \\ &- \int_0^{t^*} \left(\eta\langle\partial_t\varphi, \theta\rangle + \frac{1}{\kappa^2}\langle\nabla\hat{\varphi}, \nabla\theta\rangle + \left\langle f(\hat{\varphi}, \widehat{\mathbf{D}}), \theta\right\rangle\right) dt \\ &- \int_0^{t^*} \left(\langle\partial_t\mathbf{D}, \mathbf{W}\rangle + \left\langle \operatorname{div}\widehat{\mathbf{D}}, \operatorname{div}\mathbf{W}\right\rangle + \left\langle \operatorname{curl}\widehat{\mathbf{D}}, \operatorname{curl}\mathbf{W}\right\rangle + \left\langle \mathbf{g}\left(\hat{\varphi}, \widehat{\mathbf{D}}\right), \mathbf{W}\right\rangle\right) dt \\ &+ \int_0^{t^*} \langle\!\langle H, \mathbf{W}\cdot\boldsymbol{\tau}\rangle\!\rangle dt \\ &+ \int_0^{t^*} \left(\left\langle f\left(\hat{\varphi}, \widehat{\mathbf{D}}\right) - f(\psi, \mathbf{A}), \theta\right\rangle + \langle f_1(\psi, \mathbf{A}; \theta, \mathbf{W}), \psi - \hat{\varphi}\rangle\right) dt \\ &+ \int_0^{t^*} \left(\left\langle \mathbf{g}\left(\hat{\varphi}, \widehat{\mathbf{D}}\right) - \mathbf{g}(\psi, \mathbf{A}), \mathbf{W}\right\rangle + \left\langle \mathbf{g}_1(\psi, \mathbf{A}; \theta, \mathbf{W}), \mathbf{A} - \widehat{\mathbf{D}}\right\rangle\right) dt.\end{aligned}$$

The first five terms in (5.4) which depend solely on the discrete quantities and data will yield the desired a posteriori error estimates upon using the stability bounds in section 4. The last two terms in $\mathcal{R}$ which depend both on the discrete and continuous solutions will be shown that they are indeed of higher order and thus can be controlled by the error in the left side of (5.3) via (H4).

5.2. Residuals. We first derive the so-called Galerkin orthogonality property by rewriting the discrete problem (3.2)–(3.3) for $t^{n-1} < t \le t^n, (\omega, \mathbf{B}) \in \mathcal{H}^1(\Omega) \times \mathbf{H}^1_n(\Omega)$, and $(\widetilde{\omega}, \widetilde{\mathbf{B}}) \in \mathcal{V}^n \times \mathbf{V}^n_0$, as follows:

$$\begin{aligned}(5.5)\qquad & \eta\langle\partial_t\varphi, \omega\rangle + \frac{1}{\kappa^2}\langle\nabla\hat{\varphi}, \nabla\omega\rangle + \langle f(\varphi^n, \mathbf{D}^n), \omega\rangle \\ &= \eta\left\langle \frac{\mathcal{P}_n\varphi^{n-1} - \varphi^{n-1}}{\tau_n}, \omega\right\rangle + \langle R^n_\psi, \omega - \widetilde{\omega}\rangle + \frac{1}{\kappa^2}\langle\nabla\varphi^n, \nabla(\omega - \widetilde{\omega})\rangle,\end{aligned}$$

$$\langle \partial_t \mathbf{D}, \mathbf{B}\rangle + \left\langle \operatorname{div} \widehat{\mathbf{D}}, \operatorname{div} \mathbf{B}\right\rangle + \left\langle \operatorname{curl} \widehat{\mathbf{D}}, \operatorname{curl} \mathbf{B}\right\rangle + \langle \mathbf{P}_n[\mathbf{g}(\varphi^{n-1}, \mathbf{D}^{n-1})], \mathbf{B}\rangle$$
$$= \left\langle \frac{\mathbf{P}_n \mathbf{D}^{n-1} - \mathbf{D}^{n-1}}{\tau_n}, \mathbf{B}\right\rangle + \left\langle R^n_{\mathbf{A}}, \mathbf{B} - \widetilde{\mathbf{B}}\right\rangle + \left\langle \operatorname{div} \mathbf{D}^n, \operatorname{div}\left(\mathbf{B} - \widetilde{\mathbf{B}}\right)\right\rangle$$
$$+ \left\langle \operatorname{curl} \mathbf{D}^n, \operatorname{curl}\left(\mathbf{B} - \widetilde{\mathbf{B}}\right)\right\rangle + \left\langle\!\left\langle I^n H^n, \widetilde{\mathbf{B}} \cdot \boldsymbol{\tau}\right\rangle\!\right\rangle. \tag{5.6}$$

By taking $(\omega, \mathbf{B}) = (\theta, \mathbf{W})$ in (5.5)–(5.6) and selecting $(\widetilde{\omega}, \widetilde{\mathbf{B}})$ to be

$$\widetilde{\omega}(\cdot, t) = I^n \theta(\cdot, t), \quad \widetilde{\mathbf{B}}(\cdot, t) = I^n \mathbf{W}(\cdot, t) \quad \forall t^{n-1} < t \le t^n,$$

we then obtain an explicit expression for the residual $\mathcal{R}(\theta, \mathbf{W}) = \sum_{i=0}^{8} \mathcal{R}_i(\theta, \mathbf{W})$ of (5.4), where, for any $t^* \in (t^{m-1}, t^m]$,

$$\mathcal{R}_0(\theta, \mathbf{W}) = \eta\langle \theta^0, \psi_0 - I^0\psi_0\rangle + \langle \mathbf{W}^0, \mathbf{A}_0 - I^0 \mathbf{A}_0\rangle,$$
$$\mathcal{R}_1(\theta, \mathbf{W}) = \int_0^{t^*} \left(\eta\langle \partial_t \theta, \hat{\varphi} - \varphi\rangle + \left\langle \partial_t \mathbf{W}, \widehat{\mathbf{D}} - \mathbf{D}\right\rangle\right) dt,$$
$$\mathcal{R}_2(\theta, \mathbf{W}) = \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \left(\eta \left\langle \frac{\varphi^{n-1} - \mathcal{P}_n \varphi^{n-1}}{\tau_n}, \theta\right\rangle + \left\langle \frac{\mathbf{D}^{n-1} - \mathbf{P}_n \mathbf{D}^{n-1}}{\tau_n}, \mathbf{W}\right\rangle\right) dt,$$
$$\mathcal{R}_3(\theta, \mathbf{W}) = \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \left(\langle R^n_\psi, I^n\theta - \theta\rangle + \langle R^n_{\mathbf{A}}, I^n \mathbf{W} - \mathbf{W}\rangle\right) dt,$$
$$\mathcal{R}_4(\theta, \mathbf{W}) = \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \left(\frac{1}{\kappa^2}\langle \nabla\varphi^n, \nabla(I^n\theta - \theta)\rangle\right) dt$$
$$+ \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \left(\langle \operatorname{div} \mathbf{D}^n, \operatorname{div}(I^n \mathbf{W} - \mathbf{W})\rangle + \langle \operatorname{curl} \mathbf{D}^n, \operatorname{curl}(I^n \mathbf{W} - \mathbf{W})\rangle\right),$$
$$\mathcal{R}_5(\theta, \mathbf{W}) = \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \langle \mathbf{P}_n[\mathbf{g}(\varphi^{n-1}, \mathbf{D}^{n-1})] - \mathbf{g}(\varphi^n, \mathbf{D}^n), \mathbf{W}\rangle dt,$$
$$\mathcal{R}_6(\theta, \mathbf{W}) = \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n \wedge t^*} \left(\langle\!\langle H - I^n H^n, \mathbf{W} \cdot \boldsymbol{\tau}\rangle\!\rangle - \langle\!\langle I^n H^n, (I^n \mathbf{W} - \mathbf{W}) \cdot \boldsymbol{\tau}\rangle\!\rangle\right) dt,$$
$$\mathcal{R}_7(\theta, \mathbf{W}) = \int_0^{t^*} \left(\left\langle f\left(\hat{\varphi}, \widehat{\mathbf{D}}\right) - f(\psi, \mathbf{A}), \theta\right\rangle + \langle f_1(\psi, \mathbf{A}; \theta, \mathbf{W}), \psi - \hat{\varphi}\rangle\right) dt,$$
$$\mathcal{R}_8(\theta, \mathbf{W}) = \int_0^{t^*} \left(\left\langle \mathbf{g}\left(\hat{\varphi}, \widehat{\mathbf{D}}\right) - \mathbf{g}(\psi, \mathbf{A}), \mathbf{W}\right\rangle + \left\langle \mathbf{g}_1(\psi, \mathbf{A}; \theta, \mathbf{W}), \mathbf{A} - \widehat{\mathbf{D}}\right\rangle\right) dt.$$

Here $t^n \wedge t^* = \min(t^n, t^*)$. The rest of the arguments consists of estimating the terms $\mathcal{R}_i$ separately. For convenience, we denote by in what follows

$$\Phi^* = \| \theta^* \|_{L^2(\Omega)} + \| \mathbf{W}^* \|_{L^2(\Omega)}.$$

First by using Lemma 4.1 we easily get

$$|\mathcal{R}_0(\theta, \mathbf{W})| + |\mathcal{R}_5(\theta, \mathbf{W})| \le C\Phi^*(\mathcal{E}_0 + \mathcal{E}_1).$$

Denote by $l(t)$ the piecewise linear function $l(t) := \tau_n^{-1}(t^n - t)$; then

$$\hat{\varphi} - \varphi = l(t)(\varphi^n - \varphi^{n-1}), \quad \widehat{\mathbf{D}} - \mathbf{D} = l(t)(\mathbf{D}^n - \mathbf{D}^{n-1}) \quad \forall t^{n-1} < t \le t^n.$$

Thus we have

$$\begin{aligned}\mathcal{R}_1(\theta,\mathbf{W}) \;=\; &\sum_{n=1}^{m-1}\int_{t^{n-1}}^{t^n} l(t)\Big(\eta\langle\partial_t\theta,\varphi^n-\varphi^{n-1}\rangle+\langle\partial_t\mathbf{W},\mathbf{D}^n-\mathbf{D}^{n-1}\rangle\Big)dt\\ &+\int_{t^{m-1}}^{t^*}\frac{t^m-t}{\tau_m}\Big(\eta\langle\partial_t\theta,\varphi^m-\varphi^{m-1}\rangle+\langle\partial_t\mathbf{W},\mathbf{D}^m-\mathbf{D}^{m-1}\rangle\Big)dt\\ &:=\mathcal{R}_{1a}(\theta,\mathbf{W})+\mathcal{R}_{1b}(\theta,\mathbf{W}).\end{aligned}$$

By Corollary 4.2 we have

$$|\mathcal{R}_{1a}(\theta,\mathbf{W})|\le C\Big(\log\frac{t^*}{t^*-t^{m-1}}\Big)^{1/2}\Phi^*(\mathcal{E}_1+\mathcal{E}_2).$$

Integrating by parts and using Lemma 4.1 we get

$$\begin{aligned}\mathcal{R}_{1b}(\theta,\mathbf{W}) &= \frac{1}{\tau_m}\int_{t^{m-1}}^{t^*}\Big(\eta\langle\varphi^m-\varphi^{m-1},\theta\rangle+\langle\mathbf{D}^m-\mathbf{D}^{m-1},\mathbf{W}\rangle\Big)dt\\ &\quad+\frac{t^m-t^*}{\tau_m}\Big(\eta\langle\theta^*,\varphi^m-\varphi^{m-1}\rangle+\langle\mathbf{W}^*,\mathbf{D}^m-\mathbf{D}^{m-1}\rangle\Big)\\ &\quad-\Big(\eta\langle\theta^{m-1},\varphi^m-\varphi^{m-1}\rangle+\langle\mathbf{W}^{m-1},\mathbf{D}^m-\mathbf{D}^{m-1}\rangle\Big)\\ &\quad\le C\Phi^*\left(\|\,\varphi^m-\varphi^{m-1}\,\|_{L^2(\Omega)}+\|\,\mathbf{D}^m-\mathbf{D}^{m-1}\,\|_{L^2(\Omega)}\right).\end{aligned}$$

Thus

$$|\mathcal{R}_1(\theta,\mathbf{W})|\le C\left(1+\log\frac{t^*}{t^*-t^{m-1}}\right)^{1/2}\Phi^*(\mathcal{E}_1+\mathcal{E}_2).$$

By Lemma 4.4, the definition of the projection operator $\mathcal{P}_n:\mathcal{L}^2(\Omega)\to\mathcal{V}^n$ and the interpolation estimate (3.1), we have

$$\begin{aligned}&\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n\wedge t^*}\Big\langle\frac{\varphi^{n-1}-\mathcal{P}_n\varphi^{n-1}}{\tau_n},\theta\Big\rangle dt\\ &=\sum_{n=1}^{m}\tau_n^{-1}\langle\varphi^{n-1}-\mathcal{P}_n\varphi^{n-1},\theta^n_{\rm int}-I^n\theta^n_{\rm int}\rangle\\ &\le C\max_{1\le n\le m}\left\|\frac{h_n^2}{\tau_n}(\varphi^{n-1}-\mathcal{P}_n\varphi^{n-1})\right\|_{L^2(\Omega)}\sum_{n=1}^{m}\|\,\theta^n_{\rm int}\,\|_{H^2(\Omega)}\\ &\le C\left(1+\log\frac{t^*}{t^*-t^{m-1}}\right)^{1/2}\Phi^*\mathcal{E}_2.\end{aligned}$$

The same argument then leads to

$$|\mathcal{R}_2(\theta,\mathbf{W})|\le C\left(1+\log\frac{t^*}{t^*-t^{m-1}}\right)^{1/2}\Phi^*\mathcal{E}_2.$$

Again, by Lemma 4.4 and (3.1), we obtain

$$\begin{aligned}|\mathcal{R}_3(\theta,\mathbf{W})|&\le\sum_{n=1}^{m}\left(\|\,h_n^2R_\psi^n\,\|_{L^2(\Omega)}\|\,\theta^n_{\rm int}\,\|_{H^2(\Omega)}+\|\,h_n^2R_{\mathbf{A}}^n\,\|_{L^2(\Omega)}\|\,\mathbf{W}^n_{\rm int}\,\|_{H^2(\Omega)}\right)\\ &\le C\left(1+\log\frac{t^*}{t^*-t^{m-1}}\right)^{1/2}\Phi^*\mathcal{E}_3.\end{aligned}$$

We decompose the integral $\langle \nabla\varphi^n, \nabla(I^n\theta - \theta)\rangle$ over all elements $S \in \mathcal{M}^n$ and next integrate by parts to obtain the equivalent expression

$$\langle \nabla\varphi^n, \nabla(I^n\theta - \theta)\rangle = \sum_{e\in\mathcal{B}^n} \langle\!\langle [\![\nabla\varphi^n]\!]_e, \theta - I^n\theta\rangle\!\rangle_e + \langle\!\langle \nabla\varphi^n\cdot\mathbf{n}, I^n\theta - \theta\rangle\!\rangle,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle_e$ denotes the L^2-scalar product on $e \in \mathcal{B}^n$, and $[\![\nabla\varphi^n]\!]_e$ is defined in (3.5). In view of the interpolation estimate (3.1) we obtain

$$\sum_{n=1}^m \int_{t^{n-1}}^{t^n\wedge t^*} \langle \nabla\varphi^n, \nabla(I^n\theta - \theta)\rangle dt \le \sum_{n=1}^m |\!|\!|\, h_n^{3/2}[\![\nabla\varphi^n]\!]_e\,|\!|\!|_{L^2(\Omega)} \|\,\theta^n_{\rm int}\,\|_{H^2(\Omega)}.$$

Since $\mathbf{W}, I^n\mathbf{W} \in \mathbf{H}^1_n(\Omega)$, we have, after decomposing the integral and integrating by parts, that

$$\langle \operatorname{div}\mathbf{D}^n, \operatorname{div}(I^n\mathbf{W} - \mathbf{W})\rangle = \sum_{e\in\mathcal{B}^n} \langle\!\langle [\![\operatorname{div}\mathbf{D}^n]\!]_e, (\mathbf{W} - I^n\mathbf{W})\cdot\mathbf{n}_e\rangle\!\rangle_e, \tag{5.7}$$

$$\begin{aligned}\langle \operatorname{curl}\mathbf{D}^n, \operatorname{curl}(I^n\mathbf{W} - \mathbf{W})\rangle &= \sum_{e\in\mathcal{B}^n} \langle\!\langle [\![\operatorname{curl}\mathbf{D}^n]\!]_e, (\mathbf{W} - I^n\mathbf{W})\cdot\boldsymbol{\tau}_e\rangle\!\rangle_e \\ &+ \langle\!\langle \operatorname{curl}\mathbf{D}^n - I^nH^n, (I^n\mathbf{W} - \mathbf{W})\cdot\boldsymbol{\tau}\rangle\!\rangle + \langle\!\langle I^nH^n, (I^n\mathbf{W} - \mathbf{W})\cdot\boldsymbol{\tau}\rangle\!\rangle,\end{aligned} \tag{5.8}$$

where $\boldsymbol{\tau}_e = (-n_2^e, n_1^e)^T$ is the unit tangent of $e \in \mathcal{B}^n$. Since the last term in (5.8) cancels out a similar term in $\mathcal{R}_6$, we add $\mathcal{R}_4$ and $\mathcal{R}_6$, employ (3.1), and Lemma 4.4, and get

$$|\mathcal{R}_4(\theta, \mathbf{W}) + \mathcal{R}_6(\theta, \mathbf{W})| \le C\left(1 + \log\frac{t^*}{t^* - t^{m-1}}\right)^{1/2} \Phi^*(\mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6).$$

Now we turn to the estimate of $\Re[\mathcal{R}_7(\theta, \mathbf{W}) + \mathcal{R}_8(\theta, \mathbf{W})]$, which is different in nature. Recall that $\hat e_\psi = \psi - \hat\varphi$ and $\hat e_{\mathbf{A}} = \mathbf{A} - \widehat{\mathbf{D}}$; we obtain by simple calculations that

$$\begin{aligned}&\Re[\mathcal{R}_7(\theta, \mathbf{W}) + \mathcal{R}_8(\theta, \mathbf{W})] \\ &= \int_0^{t^*}\int_\Omega \Re\Big[\Big(\frac{\mathbf{i}}{\kappa} - \mathbf{i}\eta\kappa\Big)\operatorname{div}\hat e_{\mathbf{A}}\hat e_\psi\theta\Big]dxdt \\ &+ \int_0^{t^*}\int_\Omega \Re\left[\frac{2\mathbf{i}}{\kappa}\hat e_{\mathbf{A}}\nabla\hat e_\psi\theta\right]dxdt \\ &+ \int_0^{t^*}\int_\Omega \Re\Big[\Big(\hat e_{\mathbf{A}}\hat e_\psi\left(\mathbf{A} + \widehat{\mathbf{D}}\right) + |\hat e_{\mathbf{A}}|^2\psi + \hat e_\psi^2\bar\psi + \hat\varphi|\hat e_\psi|^2 + \psi|\hat e_\psi|^2\Big)\theta\Big] \\ &+ \int_0^{t^*}\int_\Omega \Re\Big[\frac{\mathbf{i}}{\kappa}\nabla\hat e_\psi\bar{\hat e}_\psi - 2\hat e_{\mathbf{A}}\bar{\hat e}_\psi\psi + \widehat{\mathbf{D}}|\hat e_\psi|^2\Big]\mathbf{W}dxdt \\ &:= (\mathrm{I}) + \cdots + (\mathrm{IV}).\end{aligned} \tag{5.9}$$

By Hölder inequality, (2.7), and Corollary 4.1, we get

$$\begin{aligned}|(\mathrm{I})| &\le \int_0^{t^*} \|\operatorname{div}\hat e_{\mathbf{A}}\|_{L^2(\Omega)} \|\,\hat e_\psi\,\|_{L^4(\Omega)} \|\,\theta\,\|_{L^4(\Omega)} dt \\ &\le C\left(\int_0^{t^*} \|\,\theta\,\|^4_{L^4(\Omega)} dt\right)^{1/4} \left(\int_0^{t^*} \|\operatorname{div}\hat e_{\mathbf{A}}\|^{4/3}_{L^2(\Omega)} \|\,\hat e_\psi\,\|^{2/3}_{H^1(\Omega)} \|\,\hat e_\psi\,\|^{2/3}_{L^2(\Omega)} dt\right)^{3/4}\end{aligned}$$

$$\leq C\Phi^* \max_{0\leq t\leq t^*} \| \hat{e}_\psi \|_{L^2(\Omega)}^{1/2} \left(\int_0^{t^*} \| \operatorname{div} \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)}^{4/3} \| \hat{e}_\psi \|_{H^1(\Omega)}^{2/3} dt \right)^{3/4}$$
$$\leq C\Phi^* \max_{0\leq t\leq t^*} \| \hat{e}_\psi \|_{L^2(\Omega)}^{1/2} \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{H^1(\Omega)}^{2} \right)^{1/2} \left(\int_0^{t^*} \| \hat{e}_\psi \|_{H^1(\Omega)}^{2} dt \right)^{1/4}$$
$$\leq C\Phi^* \Lambda(t^*)^{1/4} \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \right),$$

where

$$\Lambda(t^*) = \left(\int_0^{t^*} \left(\| \hat{e}_\psi \|_{H^1(\Omega)}^2 + \| \hat{e}_{\mathbf{A}} \|_{H^1(\Omega)}^2 \right) dt \right)^3 \Big/ \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)}^2 + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)}^2 \right).$$

Similarly, we have

$$|(\mathrm{II})| \leq C\Phi^* \max_{0\leq t\leq t^*} \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)}^{1/2} \left(\int_0^{t^*} \| \hat{e}_\psi \|_{H^1(\Omega)}^{2} \right)^{1/2} \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{H^1(\Omega)}^{2} dt \right)^{1/4}$$
$$\leq C\Phi^* \Lambda(t^*)^{1/4} \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \right).$$

The first term in (III) can be estimated by (4.7), Corollary 4.1, and (2.7) as follows:

$$\int_0^{t^*} \int_\Omega \Re[\hat{e}_{\mathbf{A}} \hat{e}_\psi (\mathbf{A} + \widehat{\mathbf{D}})\theta] dxdt$$
$$= \int_0^{t^*} \int_\Omega \Re[2\hat{e}_{\mathbf{A}} \hat{e}_\psi \mathbf{A}\theta - |\hat{e}_{\mathbf{A}}|^2 \hat{e}_\psi \theta] dxdt$$
$$\leq \int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{L^4(\Omega)} \| \hat{e}_\psi \|_{L^4(\Omega)} \| \mathbf{A} \|_{L^4(\Omega)} \| \theta \|_{L^4(\Omega)} dt$$
$$+ \int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{L^4(\Omega)}^2 \| e_\psi \|_{L^4(\Omega)} \| \theta \|_{L^4(\Omega)} dt$$
$$\leq C\Phi^* \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{L^4(\Omega)}^2 \| \hat{e}_\psi \|_{L^4(\Omega)}^2 dt \right)^{1/2} + C\Phi^* \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{L^4(\Omega)}^{8/3} \| \hat{e}_\psi \|_{L^4(\Omega)}^{4/3} dt \right)^{3/4}$$
$$\leq C\Phi^* \max_{0\leq t\leq t^*} \left(\| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)}^{1/2} \| \hat{e}_\psi \|_{L^2(\Omega)}^{1/2} \right) \left(\int_0^{t^*} \| \hat{e}_\psi \|_{H^1(\Omega)}^{2} \right)^{1/4} \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{H^1(\Omega)}^{2} dt \right)^{1/4}$$
$$+ C\Phi^* \max_{0\leq t\leq t^*} \left(\| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \| \hat{e}_\psi \|_{L^2(\Omega)}^{1/2} \right) \left(\int_0^{t^*} \| \hat{e}_\psi \|_{H^1(\Omega)}^{2} \right)^{1/4} \left(\int_0^{t^*} \| \hat{e}_{\mathbf{A}} \|_{H^1(\Omega)}^{2} dt \right)^{1/2}$$
$$\leq C\Phi^* \Lambda(t^*)^{1/6} \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \right)^{4/3}$$
$$+ C\Phi^* \Lambda(t^*)^{1/4} \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \right)^{3/2}.$$

The other terms in (III), (IV) can be estimated similarly to obtain

$$|(\mathrm{III})| + |(\mathrm{IV})| \leq C\Phi^* \Lambda(t^*)^{\alpha} \max_{0\leq t\leq t^*} \left(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \right)^{\beta}$$

for some $\alpha > 0$ and $\beta \geq 1$. Substituting the above estimates into (5.9) we finally obtain that

$$|\Re[\mathcal{R}_7(\theta, \mathbf{W}) + \mathcal{R}_8(\theta, \mathbf{W})]| \leq C\Phi^* \Lambda(t^*)^\alpha \max_{0\leq t\leq t^*} \Big(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \Big)^\beta$$

for some $\alpha > 0$ and $\beta \geq 1$.

5.3. Proof of Theorem 5.1. Collecting the above estimates for $\mathcal{R}_i$, and inserting them into (5.3), we obtain that

$$\begin{aligned} &\eta \| \psi(t^*) - \varphi(t^*) \|_{L^2(\Omega)} + \| \mathbf{A}(t^*) - \mathbf{D}(t^*) \|_{L^2(\Omega)} \\ &\leq C\mathcal{E}_0 + C \left(1 + \log \frac{t^*}{t^* - t^{m-1}} \right)^{1/2} \sum_{i=1}^{6} \mathcal{E}_i \\ &+ C\Lambda(t^*)^\alpha \max_{0\leq t\leq t^*} \Big(\| \hat{e}_\psi \|_{L^2(\Omega)} + \| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \Big)^\beta \end{aligned} \tag{5.10}$$

for some $\alpha > 0$ and $\beta \geq 1$. However,

$$\begin{aligned} \| \hat{e}_\psi \|_{L^2(\Omega)} &\leq \| \psi - \varphi \|_{L^2(\Omega)} + \| \varphi - \hat{\varphi} \|_{L^2(\Omega)} \\ &\leq \| \psi - \varphi \|_{L^2(\Omega)} + \max_{1\leq n\leq m} \big\| \varphi^n - \varphi^{n-1} \big\|_{L^2(\Omega)} \\ &\leq \| \psi - \varphi \|_{L^2(\Omega)} + (\mathcal{E}_1 + \mathcal{E}_2) \qquad \forall t \in (0, t^*], \end{aligned}$$

and

$$\| \hat{e}_{\mathbf{A}} \|_{L^2(\Omega)} \leq \| \mathbf{A} - \mathbf{D} \|_{L^2(\Omega)} + (\mathcal{E}_1 + \mathcal{E}_2) \qquad \forall t \in (0, t^*].$$

We use (3.4) and (H4) to conclude that for sufficiently small $h_{\max}$ and $\tau_{\max}$, the rightmost term in (5.10) can be absorbed into the left-hand side, and thus

$$\| \psi^m - \varphi^m \|_{L^2(\Omega)} + \| \mathbf{A}^m - \mathbf{D}^m \|_{L^2(\Omega)} \leq C\mathcal{E}_0 + C\Big(1 + \log \frac{t^m}{\tau_m}\Big)^{1/2} \sum_{i=1}^{6} \mathcal{E}_i.$$

This completes the proof. ∎

6. Numerical simulation. In this section we explain first how the estimators from section 5 can be used for mesh and time-step modification and then document the performance of the resulting adaptive method. In the computations we used the software "Finite Element Program Automatic Generator" by Guoping Liang and an adaptive mesh generator developed by Jian Zhang, which is based on the bisection strategy proposed in [3].

6.1. Adaptive method. We use the estimators in Theorem 5.1 to equidistribute the space contribution by refining and coarsening of the mesh $\mathcal{M}^n$ and time contribution by modifying the time-step τ_n. We take the constant C in the a posteriori error estimate equal to 1 and ignore the $\log(t^m/\tau_m)$ term. We observe that the time residual $\mathcal{E}_1^n$ and the boundary error $\mathcal{E}_6^n$ serve to adjust the τ_n; all other estimators provide information for space adaption of $\mathcal{M}^n$. For any $S \in \mathcal{M}^n$, we split the estimators $\mathcal{E}_2^n, \dots, \mathcal{E}_5^n$ into element contribution $E^n(S)$ as follows:

$$E^n(S)^2 := (1 + h_S^4 \tau_n^{-2}) \Big(\| \varphi^{n-1} - \mathcal{P}_n \varphi^{n-1} \|_{L^2(S)}^2 + \| \mathbf{D}^{n-1} - \mathbf{P}_n \mathbf{D}^{n-1} \|_{L^2(S)}^2 \Big)$$

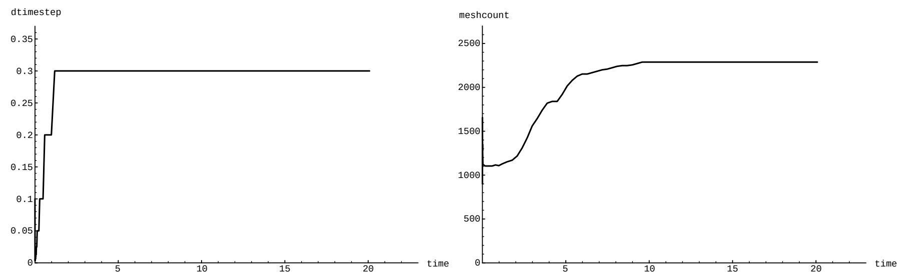

FIG. 6.1. *Time-step sizes and element counts.*

$$
\begin{aligned}
&+ h_S^4 \| R_\psi^n \|_{L^2(S)}^2 + h_S^4 \| R_{\mathbf{A}}^n \|_{L^2(S)}^2 \\
&+ \sum_{e \subset \partial S,\, e \in \mathcal{B}^n} h_e^3 \left(\| [\![\nabla \varphi^n]\!]_e \|_{L^2(e)}^2 + \| [\![\operatorname{div} \mathbf{D}^n]\!]_e \|_{L^2(e)}^2 + \| [\![\operatorname{curl} \mathbf{D}^n]\!]_e \|_{L^2(e)}^2 \right) \\
&+ \sum_{e \subset \partial S \cap \partial \Omega,\, e \in \bar{\mathcal{B}}^n} h_e^3 \left(\| \nabla \varphi^n \cdot \mathbf{n} \|_{L^2(e)}^2 + \| \operatorname{curl} \mathbf{D}^n - I^n H^n \|_{L^2(e)}^2 \right).
\end{aligned}
$$

Given mesh and time tolerance $\varepsilon_{\text{mesh}}$ and $\varepsilon_{\text{time}}$, we refine/coarsen any element $S \in \mathcal{M}^n$ according to the rules

$$
\begin{aligned}
E^n(S) &> \frac{\theta_{12}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \text{ refine } S \text{ twice; } \frac{\theta_{11}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \le E^n(S) < \frac{\theta_{12}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \text{ refine } S \text{ once,} \\
E^n(S) &< \frac{\theta_{22}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \text{ coarsen } S \text{ twice; } \frac{\theta_{22}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \le E^n(S) < \frac{\theta_{21}\varepsilon_{\text{mesh}}}{\sqrt{M_n}} \text{ coarsen } S \text{ once,}
\end{aligned}
$$

and reduce/enlarge the time-step τ_n according to the prescription

$$
\mathcal{E}_1^n + \mathcal{E}_6^n > \gamma_1 \varepsilon_{\text{time}} \quad \text{reduce } \tau_n; \qquad \mathcal{E}_1^n + \mathcal{E}_6^n < \gamma_2 \varepsilon_{\text{time}} \quad \text{enlarge } \tau_n,
$$

where $\theta_{11}, \theta_{12}, \gamma_1 \ge 1$, $\theta_{21}, \theta_{22}, \gamma_2 < 1$ are given positive constants, and M_n is the number of elements of $\mathcal{M}^n$.

6.2. Simulation. We present in this subsection several examples to illustrate the performance of the proposed adaptive method in section 6.1 for solving the TDGL model under gauge choice (1.5). In all examples we take the Ginzburg–Landau parameter $\kappa = 10$, the applied magnetic field $H = 5$, and the initial data $\psi_0 = 0.6 + \mathbf{i}\, 0.8$ and $\mathbf{A}_0 = (0,0)^T$.

In the first example we let $\Omega = (0,1) \times (0,1)$ and the length of the time interval $T = 20$. The various parameters are taken to be

$$
\begin{aligned}
&\varepsilon_{\text{mesh}} = 0.2,\ \varepsilon_{\text{time}} = 1.0,\ \gamma_1 = 1.0,\ \gamma_2 = 0.5, \\
&\theta_{11} = 2.0,\ \theta_{12} = 1.1,\ \theta_{21} = 0.4,\ \theta_{22} = 0.2.
\end{aligned}
$$

Figure 6.1 shows the number of elements in the adapted meshes $\mathcal{M}^n$ and time-step sizes. Figure 6.2 presents the adaptive meshes and the corresponding contour plots of $|\psi|^2$ at various time-steps.

In the second example we take $\Omega = (0,1.5) \times (0,1.5)$ and the length of the time interval $T = 40$. The various parameters are taken to be

$$
\begin{aligned}
&\varepsilon_{\text{mesh}} = 0.5,\ \varepsilon_{\text{time}} = 1.0,\ \gamma_1 = 1.0,\ \gamma_2 = 0.5, \\
&\theta_{11} = 2.0,\ \theta_{12} = 1.1,\ \theta_{21} = 0.4,\ \theta_{22} = 0.2.
\end{aligned}
$$

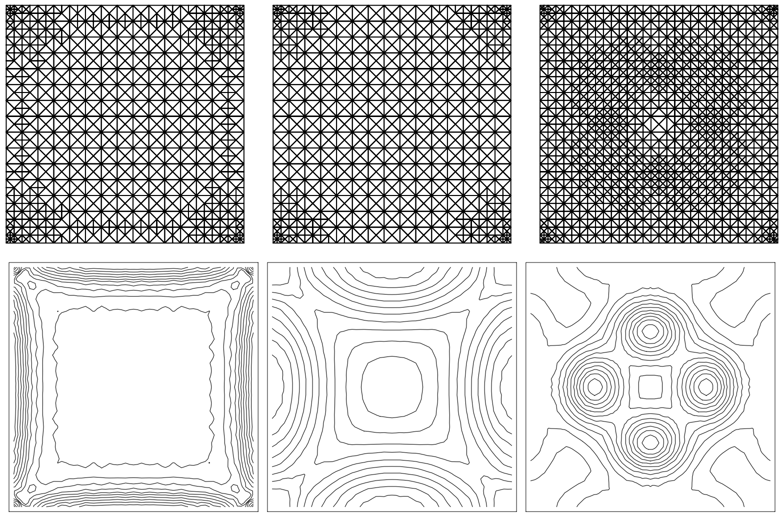

FIG. 6.2. *Contour plots and corresponding meshes at time* $t = 0.021875, 0.304, 20.0$ *(from left to right).*

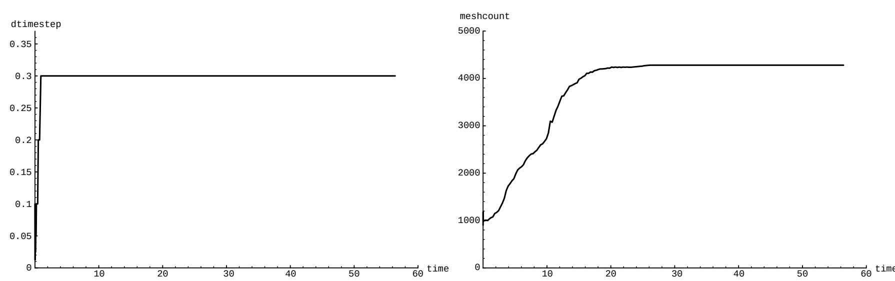

FIG. 6.3. *Time-step sizes and element counts.*

Figure 6.3 shows the number of elements in the adapted meshes $\mathcal{M}^n$ and time-step sizes. Figure 6.4 presents the adaptive meshes and the corresponding contour plots of $|\psi|^2$ at various time-steps.

We observe that the time-steps are invariant except at the beginning of the simulations because the initial state is not the stable one. In the first example the stable state is reached at $t = 13.5$, while in the second one the stable state is reached at $t = 26.0$. The adapted meshes in Figures 6.2 and 6.4 indicate clearly that our adaptive method is able to capture the motion of vortices. Another important goal of the a posteriori error analysis is to show that an "optimal mesh" is indeed generated by using the error estimators as in the spirit of adaptive methods. This highly nonlinear global optimization problem deserves further theoretical and numerical investigations. The following numerical test clearly indicates that the proposed adaptive method based on the a posteriori error estimates in this paper leads to considerable

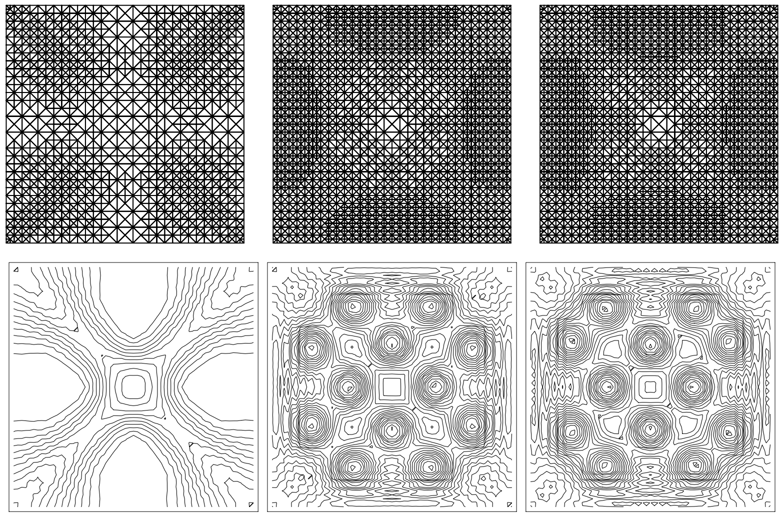

FIG. 6.4. *Contour plots and corresponding meshes at time* $t = 7.225$, 16.225, 40.0 *(from left to right).*

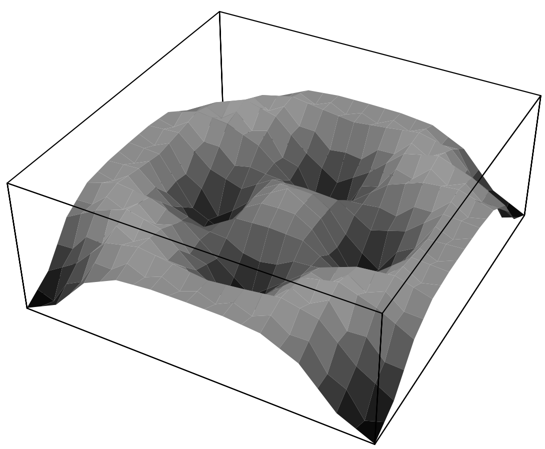

FIG. 6.5. *Surface plot of* $|\psi|^2$ *of the stable state using the uniform mesh of* 400 *elements.*

improvement in terms of the accuracy of the solutions for a given number of elements. Let $\Omega = (0,1) \times (0,1)$. Figures 6.5 and 6.6 show the surface plot of $|\psi|^2$ of the stable state using a uniform mesh of 400 elements and an adaptive mesh with 408 elements, respectively. The underlying uniform and the adaptive meshes are shown in Figure 6.7. As a comparison we show in Figure 6.8 the surface plot of $|\psi|^2$ of the stable state computed in the first example above using an adaptive mesh with 2288 elements.

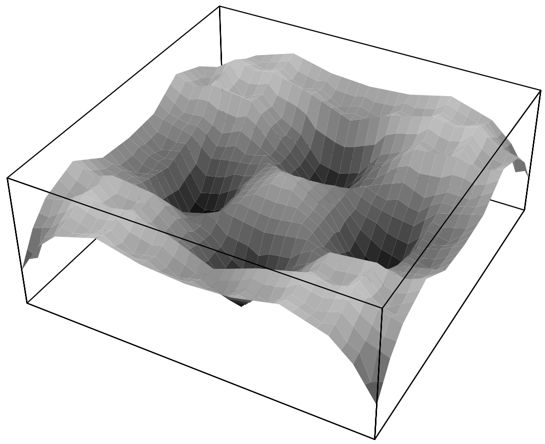

FIG. 6.6. *Surface plot of* $|\psi|^2$ *of the stable state using the adaptive mesh of* 408 *elements.*

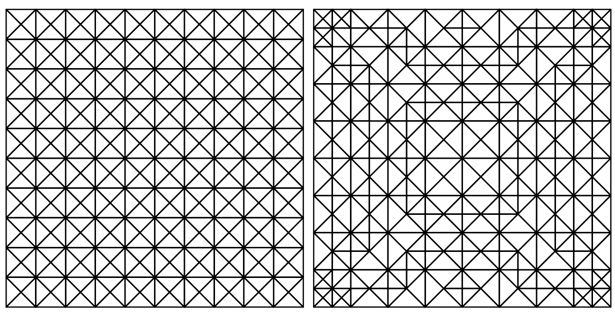

FIG. 6.7. *The uniform mesh of* 400 *elements (left) and the adaptive mesh of* 408 *elements (right).*

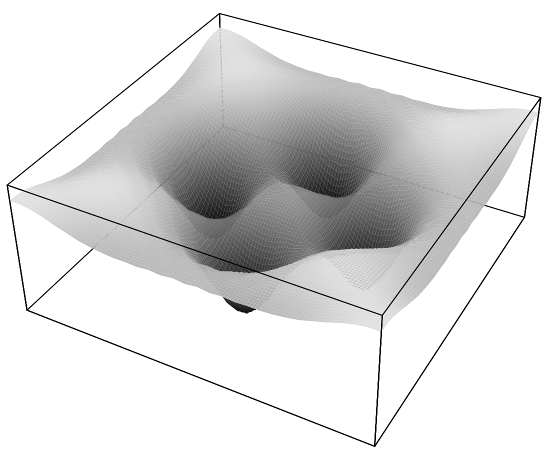

FIG. 6.8. *Surface plot of* $|\psi|^2$ *of the stable state in the first example using the adaptive mesh of* 2288 *elements.*

Acknowledgment. The authors are grateful to the referees for several valuable comments about the paper.

REFERENCES

[1] I. Babuška and A. Miller, *A feedback finite element method with a posteriori error estimation: Part* I. *The finite element method and some basic properties of a posteriori error estimator*, Comput. Methods Appl. Mech. Engrg., 61 (1987), pp. 1–40.

[2] I. Babuška and W.C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[3] E. Bänsch, *Local mesh refinement in* 2 *and* 3 *dimensions*, Impact Comput. Sci. Engrg., 3 (1991), pp. 181–191.

[4] S.J. Chapman, S.D. Howison, and J.R. Ockendon, *Macroscopic models for superconductivity*, SIAM Rev., 34 (1992), pp. 529–560.

[5] Z. Chen, *Mixed finite element methods for a dynamical Ginzburg-Landau model in superconductivity*, Numer. Math., 76 (1997), pp. 323–353.

[6] Z. Chen and K.-H. Hoffmann, *Numerical studies of a non-stationary Ginzburg-Landau model for superconductivity*, Adv. Math. Sci. Appl., 5 (1995), pp. 363–389.

[7] Z. Chen and K.-H. Hoffmann, *Optimal control of dynamical Ginzburg-Landau vortices in superconductivity*, Numer. Funct. Anal. Optim., 17 (1996), pp. 241–258.

[8] Z. Chen, K.-H. Hoffmann, and J. Liang, *On a non-stationary Ginzburg-Landau superconductivity model*, Math. Methods Appl. Sci., 16 (1993), pp. 855–875.

[9] Z. Chen, R.H. Nochetto, and A. Schmidt, *A characteristic Galerkin method with adaptive error control for the continuous casting problem*, Comput. Methods Appl. Mech. Engrg., 189 (2000), pp. 249–276.

[10] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[11] Q. Du, *Global existence and uniqueness of the time-dependent Ginzburg-Landau model for superconductivity*, Appl. Anal., 53 (1994), pp. 1–14.

[12] Q. Du, *Finite element methods for the time-dependent Ginzburg-Landau model for superconductivity*, Comput. Math. Appl., 27 (1994), pp. 119–133.

[13] Q. Du, M.D. Gunzburger, and J.S. Peterson, *Analysis and approximation of the Ginzburg-Landau model of superconductivity*, SIAM Rev., 34 (1992), pp. 54–81.

[14] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems* I: *A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[15] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems* IV: *Nonlinear problems*, SIAM J. Numer. Anal., 32 (1995), pp. 1729–1749.

[16] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1996.

[17] L.P. Gor'kov and G.M. Éliashberg, *Generalization of the L.P. Ginzburg-Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities*, Soviet Phys. JETP, 27 (1968), pp. 328–334.

[18] P. Grisvard, *Elliptic Problems on Non-smooth Domains*, Pitman, Boston, 1985.

[19] J. Ivarsson, *A Posteriori Error Analysis of a Finite Element Method for the Time-Dependent Ginzburg-Landau Equations*, preprint, Chalmers University of Technology, Göteborg, Sweden, 1998.

[20] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, RI, 1968.

[21] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal., 19 (1981), pp. 93–113.

[22] M. Mu, *A linearized Crank–Nicolson–Galerkin method for the Ginzburg–Landau model*, SIAM J. Sci. Comput., 18 (1997), pp. 1028–1039.

[23] R.H. Nochetto, *Removing the saturation assumption in a posteriori error analysis*, Istit. Lombardo Accad. Sci. Lett. Rend. A, 127 (1993), pp. 67–82.

[24] R.H. Nochetto, *Pointwise a posteriori error estimates for elliptic problems on highly graded meshes*, Math. Comp., 64 (1995), pp. 1–22.

[25] R.H. Nochetto, A. Schmidt, and C. Verdi, *A posteriori error estimation and adaptivity for degenerate parabolic problems*, Math. Comp., 69 (2000), pp. 1–24.

[26] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, 1975.