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UNIVERSAL BOUNDS ON COARSENING RATES FOR MEAN-FIELD MODELS OF PHASE TRANSITIONS*

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Abstract. We prove one-sided universal bounds on coarsening rates for two kinds of mean-field models of phase transitions, one with a coarsening rate $l \sim t^{1/3}$ and the other with $l \sim t^{1/2}$. Here l is a characteristic length scale. These bounds are both proved by following a strategy developed by Kohn and Otto [*Comm. Math. Phys.*, 229 (2002), pp. 375–395]. The $l \sim t^{1/2}$ rate is proved using a new dissipation relation which extends the Kohn–Otto method. In both cases, the dissipation relations are subtle and their proofs are based on a residual lemma (Lagrange identity) for the Cauchy–Schwarz inequality.

Key words. Ostwald ripening, Lifshitz–Slyozov–Wagner equations, scaling exponents

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1. Introduction. In the late stages of heterogeneously nucleated phase transitions, a two-phase mixture is created, composed of particles of one phase dispersed in a matrix of the other. Initially the particles are small and their total surface area is large. According to thermodynamics, the system evolves in order to decrease the total surface area and conserve the total mass or volume of the particles. Smaller particles shrink and disappear and larger ones grow. As a result, the typical length scale that characterizes the particle size increases. It is widely observed that the length scale behaves as a temporal power law.

In this paper, we will try to give this power-law behavior a rigorous mathematical explanation in the context of mean-field models. In mean-field models, particles exchange mass by some interaction through a mean field $\theta(t)$ which is determined as a function of time t by the conservation of mass. There are many mechanisms that can dominate the mass transfer process [15]. We will consider two of them in this paper that correspond to two kinds of mean-field models with different power-law behaviors.

In the first model, particle growth is controlled by bulk or volume diffusion, with or without kinetic drag at the interface. Each particle radius R obeys the growth law

$$\dot{R} = \frac{1}{R+\beta}\left(\theta(t) - \frac{1}{R}\right), \tag{1.1}$$

where $\beta \geq 0$ is a constant. The particle size distribution $f(t, R)$ satisfies the transport equation

$$\partial_t f + \partial_R \left(\frac{1}{R+\beta}\left(\theta - \frac{1}{R}\right) f \right) = 0. \tag{1.2}$$

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To conserve the total mass, the mean field θ satisfies

$$\theta(t) = \frac{\int_0^\infty (R+\beta)^{-1} R^{n-2} f(t,R)\, dR}{\int_0^\infty (R+\beta)^{-1} R^{n-1} f(t,R)\, dR}, \tag{1.3}$$

where n is the dimension of space. When $\beta = 0$, (1.1)–(1.3) is the classical model by Lifshitz and Slyozov [9] and Wagner [16]. Equation (1.1) is an approximation to the Mullins–Sekerka sharp-interface model with a modified Gibbs–Thomson law in the situation in which the particles are sparsely located in a domain Ω:

$$-\Delta u = 0 \quad \text{outside the particles}, \tag{1.4}$$

$$n \cdot \nabla u = V \quad \text{on } \Gamma, \tag{1.5}$$

$$u = \kappa + \beta V \quad \text{on } \Gamma. \tag{1.6}$$

Here Γ is the boundary of the particles, u is a chemical potential, n is the outer normal to Γ, κ is the mean curvature, and V is the normal velocity of Γ. Note that (1.6) is the Gibbs–Thomson law modified by a kinetic drag term βV. In [10, 11], Niethammer rigorously derived the model (1.1)–(1.3) in $\boldsymbol{R}^3$ for $\beta = 0$ and $\beta > 0$, respectively, from a model similar to (1.4)–(1.6) under the condition that the total capacity of the particles was small.

The second model arises formally by taking $\beta \to \infty$ after rescaling time by β. In this model, particle growth is controlled by the attachment reaction at the interface [1]. Now each particle radius R obeys the law

$$\dot{R} = \theta(t) - \frac{1}{R}. \tag{1.7}$$

The corresponding transport equation of the particle size distribution becomes

$$\partial_t f + \partial_R\left(\left(\theta - \frac{1}{R}\right) f\right) = 0. \tag{1.8}$$

In this case, the mean field θ satisfies

$$\theta(t) = \frac{\int_0^\infty R^{n-2} f(t,R)\, dR}{\int_0^\infty R^{n-1} f(t,R)\, dR}. \tag{1.9}$$

Equation (1.7) is the normalized mean curvature flow for a collection of spheres; i.e., it is a special case of the following sharp-interface model:

$$V = -\kappa + \frac{1}{|\Gamma|}\int_\Gamma \kappa\, dS, \tag{1.10}$$

where $|\Gamma|$ is the total area of the particle surface, κ is the mean curvature of the particle surface, and V is the normal velocity of the particle surface.

As in many systems, the coarsening rates for these mean-field models can be predicted by heuristic reasoning based on scaling invariance. The models we are considering are invariant under the scalings

$$R = \eta\hat{R}, \quad t = \eta^3\hat{t}, \quad \theta = \eta^{-1}\hat{\theta} \quad \text{for (1.1) if } \beta = 0; \tag{1.11}$$

$$R = \eta\hat{R}, \quad t = \eta^2\hat{t}, \quad \theta = \eta^{-1}\hat{\theta} \quad \text{for (1.7)}. \tag{1.12}$$

If one expects that over long times the behavior of the coarsening system will appear scale invariant in some rough or statistical sense, then this kind of scaling invariance suggests that a characteristic length scale $l(t)$ ought to satisfy $l(t) = \eta l(t/\eta^p)$ with $p = 3$ or 2, respectively, so that $l(t)$ will be given by a temporal power law

$$l(t) \sim t^{1/3} \quad \text{for (1.1)}, \tag{1.13}$$

$$l(t) \sim t^{1/2} \quad \text{for (1.7)}. \tag{1.14}$$

When $\beta \neq 0$, under the scaling (1.11) equation (1.1) keeps its form if β is replaced by $\widehat{\beta} = \beta/\eta$. Then (1.1) is not invariant since we assume β to be a constant. However, this suggests that as the length scale becomes large, the influence of kinetic drag can be neglected and should not influence the ultimate coarsening rate for the volume-diffusion–controlled growth model.

In three dimensions ($n = 3$), the classical Lifshitz–Slyozov–Wagner (LSW) theory suggests that the size distribution function approaches a universal self-similar solution where the critical radius $R_c = \theta^{-1}$ follows the temporal power law $R_c \sim t^{1/3}$. Such a power law is observed in experiments. However, Niethammer and Pego [12] proved that, mathematically, for solutions of (1.2) the size distribution function does not necessarily converge to the predicted universal similarity solution, and the long-time behavior need not be self-similar.

Thus, the question is whether anything can be said universally about the coarsening rate of solutions in classical LSW theory. We cannot expect all solutions to coarsen at the same rate. For example, if initially all the particles are of the same size, then the system does not coarsen at all. One would like to be able to show that the expected power-law behavior is typical in some sense. What is "typical" is not clear, but a related question is whether it is possible that some solutions coarsen faster than expected. We know of no heuristic reason that would prevent such behavior.

Recently, Kohn and Otto [6] introduced a powerful method to answer this question. They obtain rigorous, universally valid time-averaged upper bounds on coarsening rates, in the setting of Cahn–Hilliard equations, which are diffuse-interface models for phase transitions (see also [3, 7, 8] for subsequently related results). Kohn and Otto consider the standard Cahn–Hilliard equation, whose sharp-interface limit is the Mullins–Sekerka model (1.4)–(1.6) with $\beta = 0$, and the Cahn–Hilliard equation with degenerate mobility, whose sharp interface limit is the surface diffusion model. Scaling invariance suggests that these two models have coarsening rates $l \sim t^{1/3}$ and $l \sim t^{1/4}$, respectively. Define $⨍_0^T := \frac{1}{T}\int_0^T$ to indicate the time-averaged integral. The results of Kohn and Otto, in their simplest form, are estimates of the following forms:

(i) $⨍_0^T E^2(t)\, dt \geq C_2 ⨍_0^T (t^{-1/3})^2\, dt$ for $T \geq C_3 L(0)^3$ (standard Cahn–Hilliard);

(ii) $⨍_0^T E^3(t)\, dt \geq C_2 ⨍_0^T (t^{-1/4})^3\, dt$ for $T \geq C_3 L(0)^4$ (Cahn–Hilliard with degenerate mobility).

Here E is the volume-averaged free energy, which is a decreasing function of time and scales as inverse to length, L is a "length scale" that is dual to E, and C_2 and C_3 are positive constants that depend only on the dimension of space n. Thus, these estimates are time-averaged versions of $E \geq C_2 t^{-1/3}$ and $E \geq C_2 t^{-1/4}$, respectively, which correspond to upper bounds on the length scale E^{-1}. These results show that, in a time-averaged sense, it is impossible for solutions to coarsen at a rate faster than the expected power law.

Our goal in this paper is to prove universal time-averaged upper bounds on corresponding coarsening rates for the mean-field models—see (2.4) and (2.5) below.

Again, we find that no solution can coarsen at a rate faster than that expected from scaling. The mean-field models that we study have three aspects that distinguish them from those models considered in [3, 6, 7, 8]:

(i) Mean-field models concern the evolution of dilute systems; i.e., the second phase consists of only a small fraction of the whole mixture. Kohn and Otto's analysis for the Cahn–Hilliard equations breaks down in this extreme case.

(ii) There is no spatial information and hence no pattern scale in mean-field models. This requires a different definition and interpretation of the dual length scale L.

(iii) For the normalized mean curvature flow (1.10), there is no result available for the corresponding diffuse-interface model—the conserved Allen–Cahn equation (see [14] for an asymptotic analysis of this correspondence).

To handle these differences, we will need to define all relevant quantities in terms of the distribution of particle radii. For the interface-reaction–controlled model, we will establish a new dissipation relation that extends the Kohn–Otto method and enables us to prove bounds that correspond to a coarsening rate of the form $l \sim t^{1/2}$. The proof of the dissipation relations in both mean-field models requires a different technique from previous works. A key ingredient in our proofs is the use of residual lemma (Lagrange identity) for the Cauchy–Schwarz inequality to compare the dissipation rates of E and L.

2. Strategy and main results. Let us describe our strategy for obtaining bounds on coarsening rates for the mean-field models (1.2) and (1.8) and state our main results. We work at first with a collection of finitely many particles undergoing coarsening with growth laws (1.1) and (1.7), respectively, for each particle. Such a system of particles has a discrete size distribution. We will apply a strategy similar to that of Kohn and Otto [6] to get time-averaged bounds for such discrete systems, and then pass to limits in section 5 to establish the bounds for arbitrary size distributions that have finite $(n+1)$st moment.

Kohn and Otto's strategy involves two quantities that measure length scales and three key steps. The first quantity is a volume-averaged free energy or negative entropy that decreases with time and scales as inverse to length. The second quantity scales like length, but its physical interpretation is not as clear. What is important is that, in a sense to be made precise, the second quantity is dual to the first one while at the same time being controlled by it.

In our situation, thermodynamics suggests that a natural quantity that is decreasing is the surface energy, which is proportional to the total surface area S of all the particles. Analogous to the cases considered in [3, 6, 7, 8], we will consider a kind of volume average of the surface area, which gives a quantity scaling as inverse to length. Because the total volume V of the particles is conserved, it is reasonable to consider the ratio S/V. For a finite particle system, we therefore define

$$E := \frac{\sum R_i^{n-1}}{\sum R_i^n}, \tag{2.1}$$

where n is the dimension of space and the sum goes over all surviving particles. E can also be considered as the volume-weighted average of curvatures $\{1/R_i\}$. In sections 3 and 4, we will prove that E is indeed decreasing in both models considered.

We need a length scale L that is dual to E. Since radius is dual to curvature, we define L to be the volume-weighted average of the radii $\{R_i\}$, i.e.,

$$L := \frac{\sum R_i^{n+1}}{\sum R_i^n}. \tag{2.2}$$

The first step of the Kohn–Otto method is to establish an *interpolation inequality* that expresses the duality of E and L. With the definitions (2.1) and (2.2) this is easy. By the Cauchy–Schwarz inequality,

$$\sum R_i^n = \sum R_i^{(n-1)/2} R_i^{(n+1)/2} \leq \left(\sum R_i^{n-1} \sum R_i^{n+1}\right)^{1/2},$$

and this immediately yields the required interpolation inequality,

$$EL \geq 1. \tag{2.3}$$

The second step is to obtain a *dissipation inequality* that controls $\dot{L}$ in terms of $\dot{E}$. In sections 3 and 4 below, we will prove that

$$|\dot{L}|^2 \leq C_1\,(-\dot{E}) \quad \text{for volume-diffusion–controlled growth (1.1),}$$
$$|\dot{L}|^2 \leq D_1\,(-\dot{E})\,L \quad \text{for interface-reaction–controlled growth (1.7),}$$

where C_1 and D_1 are positive constants depending only on the dimension of space n. We remark that in the cases considered in [3, 6, 7, 8], the difficult part is proving the interpolation inequalities; the dissipation relations are rather easy to prove. By contrast, in the situation of the mean-field models considered here, under definitions (2.1) and (2.2) the interpolation inequality is a simple consequence of the Cauchy–Schwarz inequality and it is the dissipation relations that need careful treatment.

The third step is an ODE argument. For the case of volume-diffusion–controlled growth, Lemma 3 in [6] and the two inequalities $EL \geq 1$ and $|\dot{L}|^2 \leq C_1(-\dot{E})$ directly give us appropriate time-averaged bounds on coarsening rates. Those that involve only E, the volume-averaged surface area, take a simple form, saying that for any $1 < p < 3$ there exist positive constants C_2 and C_3, depending only on n, p, and nothing else, such that

$$\fint_0^T E(t)^p \, dt \geq C_2 \fint_0^T \left(t^{-1/3}\right)^p \, dt \quad \text{for} \quad T \geq C_3 L(0)^3. \tag{2.4}$$

This is exactly a time-averaged version of $E \geq t^{-1/3}$, which corresponds to an upper bound on the "length scale" E^{-1}.

For the case of interface-reaction–controlled growth, we will establish an ODE lemma in section 4 to show that the inequalities $EL \geq 1$ and $|\dot{L}|^2 \leq D_1 L(-\dot{E})$ give us appropriate time-averaged estimates. In particular, for any $1 < p < 2$ there exist positive constants D_2 and D_3, depending only on n, p, and nothing else, such that

$$\fint_0^T E(t)^p \, dt \geq D_2 \fint_0^T \left(t^{-1/2}\right)^p \, dt \quad \text{for} \quad T \geq D_3 L(0)^2. \tag{2.5}$$

This is a time-averaged version of $E \geq t^{-1/2}$.

Once these results for discrete systems are established, we will pass to the case of general size distributions in section 5 by applying the well-posedness and compactness results for a family of mean-field models established by Niethammer and Pego in [13].

All of our models under consideration are included in that work except for the two-dimensional (2D) volume-diffusion–controlled growth model with $\beta = 0$. Thus this case is not included in our main theorems on coarsening rates for general size distributions.

The results in [13] enable us to approximate a general distribution by a sequence of discrete ones. These results, together with an extended moment compactness result proved here in an appendix, enable us to take limits in the estimates for the discrete sequence. This leads to our main results on coarsening rates for general size distributions.

We consider such size distributions to belong to $\mathcal{P}_n$, the set of Borel probability measures on $[0,\infty)$ with finite nth moment. Topologically we regard $\mathcal{P}_n$ as a subset of the Banach space of finite Radon measures on $[0,\infty)$, which is dual to $C_0([0,\infty))$, the space of continuous functions on $[0,\infty)$ that vanish at infinity. A *measure-valued solution* of the transport equation (1.2) or (1.8) is a weak-star continuous map $t \mapsto \nu_t$ taking $[0,\infty) \to \mathcal{P}_n$ that is a solution in the sense of distributions on $(0,\infty)\times(0,\infty)$. Based on the results in [13], we will see that for each initial size distribution $\mu \in \mathcal{P}_n$, there is a unique measure-valued solution with initial value $\nu_0 = \mu$ that preserves the nth moment (total volume). The corresponding mean field is given for a.e. $t > 0$ by

$$\theta(t) = \int_0^\infty \frac{R^{n-2}}{R+\beta} d\nu_t(R) \Big/ \int_0^\infty \frac{R^{n-1}}{R+\beta} d\nu_t(R) \tag{2.6}$$

in the case of volume-diffusion–controlled growth and

$$\theta(t) = \int_0^\infty R^{n-2} d\nu_t(R) \Big/ \int_0^\infty R^{n-1} d\nu_t(R) \tag{2.7}$$

in the case of interface-reaction–controlled growth. The quantities corresponding to (2.1) and (2.2) are defined by

$$E(t) := \int_0^\infty R^{n-1}\, d\nu_t(R) \Big/ \int_0^\infty R^n\, d\nu_t(R), \tag{2.8}$$

$$L(t) := \int_0^\infty R^{n+1}\, d\nu_t(R) \Big/ \int_0^\infty R^n\, d\nu_t(R). \tag{2.9}$$

Our main results take the following form.

Theorem 2.1 (volume-diffusion–controlled growth). *Let $n \ge 2$ be an integer and $\beta \ge 0$, with $\beta > 0$ if $n = 2$, and let p be real with $1 < p < 3$. Then there exist positive constants C_2 and C_3, depending on p, n, and nothing else, such that whenever ν is a measure-valued solution of the transport equation* (1.2) *and ν_0 has finite nth and $(n+1)$st moments, we have*

$$\fint_0^T E(t)^p\, dt \ge C_2 \fint_0^T \left(t^{-1/3}\right)^p dt \quad \textit{for } T \ge C_3\, L(0)^3. \tag{2.10}$$

Theorem 2.2 (interface-reaction–controlled growth). *Let $n \ge 2$ be an integer and let p be real with $1 < p < 2$. Then there exist positive constants D_2 and D_3, depending on p, n, and nothing else, such that whenever ν is a measure-valued solution of the transport equation* (1.8) *and ν_0 has finite nth and $(n+1)$st moments, we have*

$$\fint_0^T E(t)^p\, dt \ge D_2 \fint_0^T \left(t^{-1/2}\right)^p dt \quad \textit{for } T \ge D_3\, L(0)^2. \tag{2.11}$$

3. Discrete systems I: Volume-diffusion–controlled growth. In this section, our aim is to prove the coarsening estimate (2.4) for any collection of finitely many spherical particles in $\boldsymbol{R}^n$ that undergoes coarsening controlled by volume diffusion with or without kinetic drag. The following growth law holds for each particle:

$$\dot{R}_i = \frac{1}{R_i + \beta}\left(\theta - \frac{1}{R_i}\right), \quad (1 \le i \le N(t)), \tag{3.1}$$

where R_i is the radius of the ith particle, $N(t)$ is the number of surviving particles at time t, θ is the mean field, and the dot denotes the time derivative.

By the conservation of total mass,

$$0 = \frac{d}{dt}\sum R_i^n = n\sum R_i^{n-1}\dot{R}_i = n\sum \frac{R_i^{n-1}}{R_i+\beta}\left(\theta - \frac{1}{R_i}\right). \tag{3.2}$$

Here the sum goes over all surviving particles. Thus

$$\theta = \frac{\sum (R_i+\beta)^{-1}R_i^{n-2}}{\sum (R_i+\beta)^{-1}R_i^{n-1}}. \tag{3.3}$$

The right-hand side of (3.1) is smooth as long as there is no particle disappearing. The conservation of total mass guarantees that the solution for (3.1) and (3.3) cannot blow up in finite time. So the solution is smooth and unique from time $t_0 = 0$ up to t_1 when some particles disappear. Restarting from t_1 with the remaining particles, we again get a smooth solution until a next time t_2 when some other particles disappear. In this way, we can find finitely many times $\{t_i\}$ such that the solution for (3.1) and (3.3) globally exists, is unique, and is smooth in each time interval $(t_i, t_{i+1}), i = 0, 1, \ldots$.

By definition (2.1),

$$E = \frac{\sum R_i^{n-1}}{\sum R_i^n}. \tag{3.4}$$

Notice that E is nonincreasing in time—we have

$$\begin{aligned}\dot{E} &= \frac{n-1}{\sum R_i^n}\sum R_i^{n-2}\dot{R}_i = \frac{n-1}{\sum R_i^n}\sum \frac{R_i^{n-2}}{R_i+\beta}\left(\theta - \frac{1}{R_i}\right)\\ &= \frac{n-1}{\sum R_i^n}\left[\frac{\left(\sum (R_i+\beta)^{-1}R_i^{n-2}\right)^2}{\sum (R_i+\beta)^{-1}R_i^{n-1}} - \sum \frac{R_i^{n-3}}{R_i+\beta}\right] \le 0\end{aligned} \tag{3.5}$$

since, by the Cauchy–Schwarz inequality,

$$\sum \frac{R_i^{n-2}}{R_i+\beta} = \sum\left[\frac{R_i^{(n-1)/2}}{(R_i+\beta)^{1/2}}\frac{R_i^{(n-3)/2}}{(R_i+\beta)^{1/2}}\right] \le \left(\sum \frac{R_i^{n-1}}{R_i+\beta}\sum \frac{R_i^{n-3}}{R_i+\beta}\right)^{1/2}.$$

By definition (2.2),

$$L = \frac{\sum R_i^{n+1}}{\sum R_i^n}. \tag{3.6}$$

Inequality (2.3) gives us the required interpolation inequality $EL \ge 1$. Next we establish a dissipation relation that controls $\dot{L}$ in terms of $\dot{E}$. Taking the time derivative of L, we get

$$\dot{L} = \frac{n+1}{\sum R_i^n} \sum R_i^n \dot{R}_i \;=\; \frac{n+1}{\sum R_i^n} \sum \frac{R_i^n}{R_i+\beta}\left(\theta - \frac{1}{R_i}\right)$$
$$= \frac{n+1}{\sum R_i^n}\left[\frac{\sum (R_i+\beta)^{-1}R_i^n \cdot \sum (R_i+\beta)^{-1}R_i^{n-2}}{\sum (R_i+\beta)^{-1}R_i^{n-1}} - \sum \frac{R_i^{n-1}}{R_i+\beta}\right]. \tag{3.7}$$

We can infer $\dot{L} \geq 0$ using again the Cauchy–Schwarz inequality, but we will not need this fact. We want to prove a dissipation inequality

$$|\dot{L}|^2 \leq C_1(-\dot{E}) \tag{3.8}$$

for some constant C_1 depending only on n. Choosing $C_1(n) = (n+1)^2/(n-1)$, and plugging in the expressions (3.5) and (3.7), (3.8) becomes

$$\left[\sum \frac{R_i^n}{R_i+\beta} \sum \frac{R_i^{n-2}}{R_i+\beta} - \left(\sum \frac{R_i^{n-1}}{R_i+\beta}\right)^2\right]^2$$
$$\leq \sum R_i^n \sum \frac{R_i^{n-1}}{R_i+\beta} \cdot \left[\sum \frac{R_i^{n-1}}{R_i+\beta} \sum \frac{R_i^{n-3}}{R_i+\beta} - \left(\sum \frac{R_i^{n-2}}{R_i+\beta}\right)^2\right]. \tag{3.9}$$

Lemma 3.1. *Inequality* (3.9) *holds for any sequence of positive numbers* $\{R_i\}_{i=1}^N$.

To prove Lemma 3.1, we need the following lemma from [2].

Lemma 3.2 (Lagrange identity).

$$\left(\sum_{i=1}^N x_i^2\right)\left(\sum_{i=1}^N y_i^2\right) - \left(\sum_{i=1}^N x_i y_i\right)^2 = \sum_{\substack{i,j=1\\ i<j}}^N (x_i y_j - x_j y_i)^2 \tag{3.10}$$

for any sequences of real numbers $\{x_i\}_{i=1}^N$ *and* $\{y_i\}_{i=1}^N$.

Proof of Lemma 3.1. The proof consists of several careful applications of the Lagrange identity and the Cauchy–Schwarz inequality. Taking $x_i = (R_i^n/(R_i+\beta))^{1/2}$ and $y_i = (R_i^{n-2}/(R_i+\beta))^{1/2}$ in (3.10), we get

$$I := \sum \frac{R_i^n}{R_i+\beta} \sum \frac{R_i^{n-2}}{R_i+\beta} - \left(\sum \frac{R_i^{n-1}}{R_i+\beta}\right)^2 \tag{3.11}$$
$$= \sum_{\substack{i,j=1\\ i<j}}^N \left[\left(\frac{R_i^n}{R_i+\beta}\right)^{1/2}\left(\frac{R_j^{n-2}}{R_j+\beta}\right)^{1/2} - \left(\frac{R_j^n}{R_j+\beta}\right)^{1/2}\left(\frac{R_i^{n-2}}{R_i+\beta}\right)^{1/2}\right]^2$$
$$= \sum_{\substack{i,j=1\\ i<j}}^N \frac{R_i^{n-2}R_j^{n-2}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2$$
$$\leq \left\{\sum_{\substack{i,j=1\\ i<j}}^N \frac{R_i^{n-1}R_j^{n-1}(R_i-R_j)^2}{(R_i+\beta)(R_j+\beta)}\right\}^{1/2} \cdot \left\{\sum_{\substack{i,j=1\\ i<j}}^N \frac{R_i^{n-3}R_j^{n-3}(R_i-R_j)^2}{(R_i+\beta)(R_j+\beta)}\right\}^{1/2}.$$

Taking $x_i = R_i^{n/2}$ and $y_i = (R_i^{n-1}/(R_i+\beta))^{1/2}$ in (3.10), we get

$$\sum R_i^n \sum \frac{R_i^{n-1}}{R_i+\beta} \tag{3.12}$$

$$= \left[\sum\left(\frac{R_i^{2n-1}}{R_i+\beta}\right)^{1/2}\right]^2 + \sum_{\substack{i,j=1\\ i<j}}^{N}\left[R_i^{n/2}\left(\frac{R_j^{n-1}}{R_j+\beta}\right)^{1/2} - R_j^{n/2}\left(\frac{R_i^{n-1}}{R_i+\beta}\right)^{1/2}\right]^2$$
$$\geq \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-1}R_j^{n-1}}{(R_i+\beta)(R_j+\beta)}\Big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\Big]^2.$$

Taking $x_i = (R_i^{n-1}/(R_i+\beta))^{1/2}$ and $y_i = (R_i^{n-3}/(R_i+\beta))^{1/2}$ in (3.10), we get

$$(3.13)\quad \sum\frac{R_i^{n-1}}{R_i+\beta}\sum\frac{R_i^{n-3}}{R_i+\beta} - \left(\sum\frac{R_i^{n-2}}{R_i+\beta}\right)^2$$
$$= \sum_{\substack{i,j=1\\ i<j}}^{N}\left[\left(\frac{R_i^{n-1}}{R_i+\beta}\right)^{1/2}\left(\frac{R_j^{n-3}}{R_j+\beta}\right)^{1/2} - \left(\frac{R_j^{n-1}}{R_j+\beta}\right)^{1/2}\left(\frac{R_i^{n-3}}{R_i+\beta}\right)^{1/2}\right]^2$$
$$= \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-3}R_j^{n-3}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2.$$

Thus

$$(3.14)\qquad II := \sum R_i^n \sum\frac{R_i^{n-1}}{R_i+\beta}\left[\sum\frac{R_i^{n-1}}{R_i+\beta}\sum\frac{R_i^{n-3}}{R_i+\beta} - \left(\sum\frac{R_i^{n-2}}{R_i+\beta}\right)^2\right]$$
$$\geq \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-1}R_j^{n-1}}{(R_i+\beta)(R_j+\beta)}\Big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\Big]^2$$
$$\cdot \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-3}R_j^{n-3}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2.$$

Comparing (3.11) and (3.14), $I^2 \leq II$ is an immediate consequence of the inequality

$$(3.15)\qquad (R_i-R_j)^2 \leq \big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\big]^2 \quad \text{for all } i,j.$$

Inequality (3.15) holds since

$$\big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\big]^2 - (R_i-R_j)^2$$
$$= \beta(R_i+R_j) + 2R_iR_j - 2R_i^{1/2}R_j^{1/2}(R_i+\beta)^{1/2}(R_j+\beta)^{1/2},$$

and

$$\big[\beta(R_i+R_j)+2R_iR_j\big]^2 - \big[2R_i^{1/2}R_j^{1/2}(R_i+\beta)^{1/2}(R_j+\beta)^{1/2}\big]^2$$
$$= \beta^2(R_i-R_j)^2 \geq 0. \qquad \square$$

The dissipation inequality (3.8) follows from Lemma 3.1. Applying Lemma 3 in [6], we directly get the following estimates.

THEOREM 3.3. *For any* $0 \leq \lambda \leq 1$ *and* $0 < r < 3$ *satisfying* $\lambda r > 1$ *and* $(1-\lambda)r < 2$, *there exist positive constants* C_2 *and* C_3, *depending only on* λ, r, *and the dimension of space* n, *such that for any solution* $\{R_i\}$ *of equations* (3.1) *and* (3.3), *we have*

$$(3.16)\qquad ⨍_0^T E^{\lambda r}L^{-(1-\lambda)r}\,dt \geq C_2 ⨍_0^T \left(t^{-1/3}\right)^r dt \quad \text{for} \quad T \geq C_3\, L(0)^3,$$

where E and L are defined in terms of (2.1) *and* (2.2), *respectively.*

Proof. Lemma 3.1 guarantees that the dissipation relation (3.8) holds. Together with the interpolation inequality (2.3), we have

$$EL \geq 1 \quad \text{and} \quad |\dot{L}|^2 \leq C_1(-\dot{E}).$$

Lemma 3 in [6] leads directly to (3.3). ◻

Taking $\lambda = 1$ and $r = p$ for $1 < p < 3$ in Theorem 3.3 yields (2.4).

4. Discrete systems II: Interface-reaction–controlled growth. Our aim in this section is to prove the coarsening estimate (2.5) for any collection of finitely many spherical particles in $\boldsymbol{R}^n$ that undergoes coarsening controlled by interface reactions. Each particle obeys the growth law

$$\dot{R}_i = \theta - \frac{1}{R_i}, \quad (1 \leq i \leq N(t)), \tag{4.1}$$

where R_i is the radius of the ith particle and θ is the mean field.

By the conservation of total mass,

$$0 = \frac{d}{dt}\sum R_i^n = n\sum R_i^{n-1}\dot{R}_i = n\sum R_i^{n-1}\left(\theta - \frac{1}{R_i}\right), \tag{4.2}$$

and thus

$$\theta = \frac{\sum R_i^{n-2}}{\sum R_i^{n-1}}. \tag{4.3}$$

Solutions of the system (4.1) and (4.3) have the same global existence and piecewise smooth properties as that of (3.1) and (3.3). Taking the time derivative of $E = \sum R_i^{n-1}/\sum R_i^n$, we have

$$\begin{aligned} \dot{E} &= \frac{n-1}{\sum R_i^n}\sum R_i^{n-2}\dot{R}_i = \frac{n-1}{\sum R_i^n}\sum R_i^{n-2}\left(\theta - \frac{1}{R_i}\right) \\ &= \frac{n-1}{\sum R_i^n}\left[\frac{\left(\sum R_i^{n-2}\right)^2}{\sum R_i^{n-1}} - \sum R_i^{n-3}\right] \leq 0, \end{aligned} \tag{4.4}$$

since

$$\sum R_i^{n-2} = \sum\left[R_i^{(n-1)/2}R_i^{(n-3)/2}\right] \leq \left(\sum R_i^{n-1}\right)^{1/2}\left(\sum R_i^{n-3}\right)^{1/2}. \tag{4.5}$$

Taking the time derivative of $L = \sum R_i^{n+1}/\sum R_i^n$, we have

$$\dot{L} = \frac{n+1}{\sum R_i^n}\sum R_i^n\dot{R}_i = \frac{n+1}{\sum R_i^n}\left[\frac{\sum R_i^n\sum R_i^{n-2}}{\sum R_i^{n-1}} - \sum R_i^{n-1}\right]. \tag{4.6}$$

Again, by the Cauchy–Schwarz inequality, we can infer $\dot{L} \geq 0$.

As described in section 2, we will need a dissipation inequality that relates $\dot{L}$ and $\dot{E}$. We claim that

$$|\dot{L}|^2 \leq D_1 L(-\dot{E}) \tag{4.7}$$

for some positive constant D_1 depending only on n. Choosing $D_1(n) = (n+1)^2/(n-1)$, and plugging in the expressions (4.4) and (4.6), inequality (4.7) becomes

$$\left[\sum R_i^n \sum R_i^{n-2} - \left(\sum R_i^{n-1}\right)^2\right]^2 \le \sum R_i^{n-1} \sum R_i^{n+1} \cdot \left[\sum R_i^{n-1} \sum R_i^{n-3} - \left(\sum R_i^{n-2}\right)^2\right]. \tag{4.8}$$

LEMMA 4.1. *Inequality* (4.8) *holds for any sequence of positive numbers* $\{R_i\}_{i=1}^N$.

Proof. Similar to the proof of Lemma 3.1, we will apply the Lagrange identity (3.10) and the Cauchy–Schwarz inequality. Taking $x_i = R_i^{n/2}$ and $y_i = R_i^{(n-2)/2}$ in (3.10), we have

$$\begin{aligned} I &:= \sum R_i^n \sum R_i^{n-2} - \left(\sum R_i^{n-1}\right)^2 \\ &= \sum_{\substack{i,j=1 \\ i<j}}^N \left[R_i^{n/2} R_j^{(n-2)/2} - R_j^{n/2} R_i^{(n-2)/2}\right]^2 \\ &= \sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-2} R_j^{n-2} (R_i - R_j)^2 \\ &\le \left[\sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_i - R_j)^2\right]^{1/2} \left[\sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_i - R_j)^2\right]^{1/2}. \end{aligned} \tag{4.9}$$

Taking $x_i = R_i^{(n-1)/2}$ and $y_i = R_i^{(n+1)/2}$ in (3.10), we have

$$\begin{aligned} \sum R_i^{n-1} \sum R_i^{n+1} &= \left(\sum R_i^n\right)^2 + \sum_{\substack{i,j=1 \\ i<j}}^N \left(R_i^{(n-1)/2} R_j^{(n+1)/2} - R_j^{(n-1)/2} R_i^{(n+1)/2}\right)^2 \\ &= \left(\sum R_i^n\right)^2 + \sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_j - R_i)^2. \end{aligned} \tag{4.10}$$

Taking $x_i = R_i^{(n-1)/2}$ and $y_i = R_i^{(n-3)/2}$ in (3.10), we have

$$\begin{aligned} \sum R_i^{n-1} \sum R_i^{n-3} - \left(\sum R_i^{n-2}\right)^2 &= \sum_{\substack{i,j=1 \\ i<j}}^N \left[R_i^{(n-1)/2} R_j^{(n-3)/2} - R_j^{(n-1)/2} R_i^{(n-3)/2}\right]^2 \\ &= \sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_i - R_j)^2. \end{aligned} \tag{4.11}$$

Thus

$$\begin{aligned} II &:= \sum R_i^{n-1} \sum R_i^{n+1} \left[\sum R_i^{n-1} \sum R_i^{n-3} - \left(\sum R_i^{n-2}\right)^2\right] \\ &\ge \sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_j - R_i)^2 \sum_{\substack{i,j=1 \\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_j - R_i)^2 \\ &\ge I^2. \qquad \square \end{aligned} \tag{4.12}$$

At this point we have established the desired interpolation and dissipation inequalities. The third step toward our coarsening estimates is an ODE lemma.

LEMMA 4.2 (ODE lemma). *Let $E(t)$ and $L(t)$ be two continuous and piecewise smooth positive functions. Assume that for some T_1, $0 \le T_1 \le \infty$, $E(t)$ satisfies*

$$\dot{E} < 0 \text{ a.e. on } (0, T_1), \quad \dot{E} = 0 \text{ on } (T_1, \infty). \tag{4.13}$$

If $E(t)$ and $L(t)$ satisfy

$$EL \ge 1 \quad \text{and} \quad |\dot{L}|^2 \le D_1 L(-\dot{E}), \tag{4.14}$$

then for any $0 \le \lambda \le 1$ and $r > 0$ satisfying

$$r < 3, \quad \lambda r > 1 \quad \text{and} \quad (1-\lambda) r < 2, \tag{4.15}$$

we have

$$⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \ge D_2 ⨍_0^T (t^{-1/2})^{r-1}\, dt \;\text{ for }\; T \ge D_3 L(0)^2, \tag{4.16}$$

where D_2 and D_3 are positive constants depending only on λ, r, and D_1.

We remark that this lemma is key for obtaining bounds on coarsening rates for the $t^{1/2}$ growth law. We will extend the ideas in the proof of Lemma 3 in [6] to establish this result. A special case of Lemma 4.2 is to take $r = p + 1$ and $\lambda = p/(p + 1)$ for $1 < p < 2$. In this case, we obtain (2.5)

$$⨍_0^T E(t)^p\, dt \ge D_2 ⨍_0^T (t^{-1/2})^p\, dt \;\text{ for }\; T \ge D_3 L(0)^2, \tag{4.17}$$

where D_2 and D_3 are positive constants depending only on p and D_1.

Proof of Lemma 4.2. (1) If $T_1 = 0$, then $\dot{E} = 0$ on $(0, \infty)$. By assumption (4.14), we get $\dot{L} = 0$ on $(0, \infty)$. Hence $E(t) = E(0)$ and $L(t) = L(0)$ for all $t \in (0, \infty)$. By (4.15), $\lambda r > 1$ and $0 \le \lambda \le 1$ imply that $r > 1/\lambda \ge 1$. Hence we have $1 < r < 3$. Then

$$\begin{aligned} ⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt &= E(0)^{\lambda r} L(0)^{1-(1-\lambda)r} \\ &\ge L(0)^{1-r} \\ &\ge \frac{2}{3-r} T^{(1-r)/2} \qquad \text{if } T \ge \left(\frac{2}{3-r}\right)^{2/(r-1)} L(0)^2 \\ &= D_2' ⨍_0^T (t^{-1/2})^{r-1}\, dt \qquad \text{if } T \ge D_3' L(0)^2, \end{aligned} \tag{4.18}$$

where

$$D_2' = 1 \quad \text{and} \quad D_3' = \left(\frac{2}{3-r}\right)^{2/(r-1)}.$$

(2) Now we consider the case when $T_1 > 0$. $\dot{E}(t) < 0$ on $(0, T_1)$ implies that E is a strictly decreasing function of t on $(0, T_1)$. Hence $E(t)$ is invertible on $(0, T_1)$ and we regard $t \in (0, T_1)$ as a function of ε, with ε denoting the independent variable to distinguish it from $E = E(t)$ and avoid confusion. Note that ε ranges from $E(0)$ to

$E(\infty) := \lim_{t\to\infty} E(t)$, since $\dot{E}(t) = 0$ for $t \in (T_1, \infty)$ implies that $E(t) = E(T_1)$ for any $t > T_1$. Consequently, for $t \in (0, T_1)$, $L(t)$ can be viewed as a function of ε. Thus

$$\frac{dL}{dt} = \frac{dL}{d\varepsilon}\frac{dE}{dt} \quad \text{for } t \in (0, T_1) \tag{4.19}$$

and $|\dot{L}|^2 \le D_1 L(-\dot{E})$ implies that

$$\left|\frac{dL}{d\varepsilon}\right|^2 (-\dot{E}) \le D_1 L(\varepsilon). \tag{4.20}$$

Multiplying both sides by a positive function $f(\varepsilon)$ and integrating from 0 to T, we have

$$\int_0^T f(E(t))L(t)\,dt \ge \frac{1}{D_1}\int_{E_T}^{E_0} f(\varepsilon)\left(\frac{dL}{d\varepsilon}\right)^2 d\varepsilon \tag{4.21}$$

if $T < T_1$, and

$$\int_0^T f(E(t))L(t)\,dt \ge \int_0^{T_1} f(E(t))L(t)\,dt \ge \frac{1}{D_1}\int_{E_T}^{E_0} f(\varepsilon)\left(\frac{dL}{d\varepsilon}\right)^2 d\varepsilon \tag{4.22}$$

if $T \ge T_1$, where $E_0 = E(0)$ and $E_T = E(T)$. Taking $f(\varepsilon) = \varepsilon^{\lambda r} L(\varepsilon)^{-(1-\lambda)r}$, we get

$$\int_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\,dt \ge \frac{1}{D_1}\int_{E_T}^{E_0} \varepsilon^{\lambda r} L(\varepsilon)^{-(1-\lambda)r}\left(\frac{dL}{d\varepsilon}\right)^2 d\varepsilon. \tag{4.23}$$

We will change variables so that the right-hand side becomes an integral of a square of some gradient. Consider

$$\hat{\varepsilon} = \frac{1}{\lambda r - 1}\varepsilon^{-(\lambda r - 1)}, \quad \hat{L} = \frac{1}{1 - r(1-\lambda)/2} L^{1-r(1-\lambda)/2}. \tag{4.24}$$

Our requirements $\lambda r > 1$ and $(1-\lambda)r < 2$ guarantee that $\hat{\varepsilon} > 0, \hat{L} > 0$ and $\hat{\varepsilon} \to \infty, \hat{L} \to \infty$ as $\varepsilon \to 0$ and $L \to \infty$, respectively. Also, we have

$$\frac{d\hat{\varepsilon}}{d\varepsilon} = -\varepsilon^{-\lambda r} \tag{4.25}$$

and

$$\left(\frac{d\hat{L}}{d\hat{\varepsilon}}\right)^2 d\hat{\varepsilon} = \left(\frac{d\hat{L}}{dL}\right)^2\left(\frac{dL}{d\varepsilon}\right)^2\left(\frac{d\varepsilon}{d\hat{\varepsilon}}\right)^2\left(\frac{d\hat{\varepsilon}}{d\varepsilon}\right) d\varepsilon = -\varepsilon^{\lambda r} L^{-(1-\lambda)r}\left(\frac{dL}{d\varepsilon}\right)^2 d\varepsilon. \tag{4.26}$$

Hence

$$\int_{E_T}^{E_0} \varepsilon^{\lambda r} L^{-(1-\lambda)r}\left(\frac{dL}{d\varepsilon}\right)^2 d\varepsilon = \int_{\hat{E}_0}^{\hat{E}_T}\left(\frac{d\hat{L}}{d\hat{\varepsilon}}\right)^2 d\hat{\varepsilon}. \tag{4.27}$$

The right-hand side is bounded below by its minimum over all functions $\hat{L}(\hat{\varepsilon})$ with the same endpoint values

$$\hat{L}_0 := \hat{L}(\hat{E}_0) = \frac{1}{1 - r(1-\lambda)/2} L(0)^{1-r(1-\lambda)/2}$$

and

$$\hat{L}_T := \hat{L}(\hat{E}_T) = \frac{1}{1 - r(1-\lambda)/2} L(t)^{1-r(1-\lambda)/2},$$

and the minimum is achieved when $\hat{L}$ is a linear function of $\hat{\varepsilon}$. Thus

$$\int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt \geq \frac{1}{D_1} \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_T - \hat{E}_0}. \tag{4.28}$$

(2a) If $L(T) \geq 2L(0)$, then

$$\hat{L}_0 \leq 2^{r(1-\lambda)/2-1} \hat{L}_T < \hat{L}_T.$$

Hence

$$\hat{L}_T - \hat{L}_0 \geq \left(1 - 2^{r(1-\lambda)/2-1}\right) \hat{L}_T$$

and, consequently,

$$\begin{aligned} \int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt &\geq \frac{1}{D_1} \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_T - \hat{E}_0} \geq \frac{\left(1 - 2^{r(1-\lambda)/2-1}\right)^2}{D_1} \frac{\hat{L}_T^2}{\hat{E}_T} \\ &\geq \frac{\left(1 - 2^{r(1-\lambda)/2-1}\right)^2}{D_1} \frac{(\lambda r - 1)}{(1 - r(1-\lambda)/2)^2}\, E^{\lambda r - 1} L^{2-(1-\lambda)r} \\ &= \hat{D}_2\, (EL)^{((2\lambda-1)r+1)/(r-1)} \cdot \left(E^{\lambda r} L^{1-(1-\lambda)r}\right)^{-(r-3)/(1-r)}, \end{aligned} \tag{4.29}$$

where

$$\hat{D}_2 := \frac{(\lambda r - 1)}{D_1} \left(\frac{1 - 2^{r(1-\lambda)/2-1}}{1 - r(1-\lambda)/2} \right)^2.$$

Since $1 < r < 3$ and $\lambda r > 1$,

$$(2\lambda - 1)r + 1 = 2\lambda r + 1 - r > 3 - r > 0.$$

Thus

$$\frac{(2\lambda-1)r+1}{r-1} > 0 \quad \text{and} \quad \frac{r-3}{1-r} > 0.$$

So $EL \geq 1$ implies $(EL)^{((2\lambda-1)r+1)/(r-1)} \geq 1$ and hence

$$\int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt \geq \hat{D}_2 \left(E^{\lambda r} L^{1-(1-\lambda)r}\right)^{-(r-3)/(1-r)} \quad \text{if } L_T \geq 2L_0. \tag{4.30}$$

Define

$$h(T) := \int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt. \tag{4.31}$$

Then $h'(T) = E^{\lambda r}(T) L^{1-(1-\lambda)r}(T)$ and (4.30) can be rewritten as

$$h(T) \geq \hat{D}_2 (h'(T))^{-(r-3)/(1-r)} \quad \text{if } L_T \geq 2L_0 \tag{4.32}$$

or, equivalently,

$$h'(T)\big(h(T)\big)^{(r-1)/(3-r)} \ge \hat{D}_2^{(r-1)/(3-r)} \quad \text{if } L_T \ge 2L_0. \tag{4.33}$$

(2b) If $L(T) < 2L(0)$, then by $EL \ge 1$,

$$E_T \ge L_T^{-1} \ge \frac{1}{2}L_0^{-1},$$

$$E_T^{\lambda r} L_T^{1-(1-\lambda)r} = \Big(E_T L_T\Big)^{\lambda r} L_T^{1-r} \ge L_T^{1-r} \ge L_0^{1-r}\, 2^{1-r}.$$

Thus

$$h'(T) \ge L_0^{1-r}\, 2^{1-r} \quad \text{if } \ L(T) \le 2L(0). \tag{4.34}$$

Combining (4.33) and (4.34), we have

$$h'(T)\Big[h(T) + L_0^{3-r}\Big]^{(r-1)/(3-r)} \ge \min\{2^{1-r}, \hat{D}_2^{(r-1)/(3-r)}\} =: m \quad \text{for all } T.$$

Thus

$$\frac{d}{dt}\Big[h(t) + L_0^{3-r}\Big]^{2/(3-r)} = \frac{2}{3-r}h'(t)\Big[h(t) + L_0^{3-r}\Big]^{(r-1)/(3-r)} \ge \frac{2m}{3-r}. \tag{4.35}$$

By integration over time from 0 to T, we get

$$\begin{aligned} h(T) &\ge \left[\frac{2m}{3-r}T + L_0^2\right]^{(3-r)/2} - L_0^{3-r} \\ &\ge \left(\frac{2m}{3-r}\right)^{(3-r)/2} T^{(3-r)/2} - L_0^{3-r} \\ &\ge \frac{1}{2}\left(\frac{2m}{3-r}\right)^{(3-r)/2} T^{(3-r)/2} \quad \text{if } \ T > 2^{2/(3-r)}\frac{(3-r)}{2m}L_0^2. \end{aligned} \tag{4.36}$$

Equivalently,

$$\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \ge D_2'' \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T > D_3'' L_0^2, \tag{4.37}$$

where

$$D_2'' = \frac{3-r}{4}\left(\frac{2m}{3-r}\right)^{(3-r)/2} \quad \text{and} \quad D_3'' = 2^{2/(3-r)}\frac{(3-r)}{2m}.$$

(3) Combining (1) and (2), we conclude that

$$\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \ge D_2 \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T > D_3 L_0^2, \tag{4.38}$$

where

$$D_2 = \min\{D_2', D_2''\} \quad \text{and} \quad D_3 = \max\{D_3', D_3''\}. \qquad \square$$

We claim the following estimate for the collection of particles that undergoes coarsening determined by (4.1).

THEOREM 4.3. *For any* $0 \le \lambda \le 1$ *and* $0 < r < 3$ *satisfying* $\lambda r > 1$ *and* $(1-\lambda)r < 2$, *there exist positive constants* D_2 *and* D_3, *depending only on* λ, r, *and the dimension of space* n, *such that for any solution* $\{R_i\}$ *of equations* (4.1) *and* (4.3), *we have*

$$\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \ge D_2 \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T \ge D_3 L(0)^2, \tag{4.39}$$

where E *and* L *are defined in terms of* (2.1) *and* (2.2), *respectively.*

Proof. As we discussed at the beginning of this section, solutions $\{R_i\}$ of equations (4.1) and (4.3) are continuous and piecewise smooth. Hence E and L defined by (2.1) and (2.2) are continuous and piecewise smooth. Furthermore, by (4.4), $\dot{E} \le 0$ and $\dot{E} = 0$ if and only if all R_i are equal. Notice that if all R_i are equal, then the system (4.1) and (4.3) reaches an equilibrium point and the solution stops coarsening. Consequently, if $\dot{E} = 0$ at some time t_1, then $\dot{E}(t) = 0$ for all $t \ge t_1$. Hence, $\dot{E}$ satisfies the condition (4.13) of Lemma 4.2.

On the other hand, the interpolation inequality (2.3) and the dissipation relation (4.7) say

$$EL \ge 1 \quad \text{and} \quad |\dot{L}|^2 \le D_1 L(-\dot{E}).$$

The theorem is then an immediate consequence of Lemma 4.2. □

5. Coarsening rates for particle systems with general size distributions. Now it is time to consider our mean-field models with more general size distributions. Definitions (2.1) and (2.2) imply that, in the more general case, E and L should be defined in terms of the $(n-1)$st, nth, and $(n+1)$st moments of the size distributions. Thus it is necessary to require the initial size distributions to be in $\mathcal{P}_{n+1}$, the set of Borel probability measures on $[0,\infty)$ with finite $(n+1)$st moments. By Hölder's inequality, it is immediate to see that $\mathcal{P}_{n+1}$ is a subset of $\mathcal{P}_n$.

In [13], Niethammer and Pego proved well-posedness and compactness results for a family of mean-field models. All of our models under consideration are included in that work except for the 2D volume-diffusion–controlled growth model with $\beta = 0$. Their results guarantee the existence and uniqueness of measure-valued solutions of equation (1.2) or (1.8). A measure-valued solution is a weak-star continuous map $t \mapsto \nu_t$ taking $[0,\infty) \to \mathcal{P}_n$ that is a solution in the sense of distributions on $(0,\infty)\times(0,\infty)$; i.e., for all $\phi \in C_c^\infty([0,\infty)\times(0,\infty))$ (smooth functions with compact support),

$$\int_0^\infty \int_0^\infty \left(\partial_t\phi + \frac{1}{R+\beta}\left(\theta(t) - \frac{1}{R}\right)\partial_R\phi\right) d\nu_t\, dt + \int_0^\infty \phi(0,\cdot)\, d\nu_0 = 0 \tag{5.1}$$

in the case of volume-diffusion–controlled growth (1.2), or

$$\int_0^\infty \int_0^\infty \left(\partial_t\phi + \left(\theta(t) - \frac{1}{R}\right)\partial_R\phi\right) d\nu_t\, dt + \int_0^\infty \phi(0,\cdot)\, d\nu_0 = 0 \tag{5.2}$$

in the case of interface-reaction–controlled growth (1.8).

Our main results are estimates in terms of these measure-valued solutions.

THEOREM 5.1 (volume-diffusion–controlled growth). *Let* $n \ge 2$ *be an integer and* $\beta \ge 0$, *with* $\beta > 0$ *if* $n = 2$. *For any* $0 \le \lambda \le 1$ *and* $0 < r < 3$ *satisfying* $\lambda r > 1$ *and* $(1-\lambda)r < 2$, *there exist positive constants* C_2 *and* C_3, *depending only on* λ, r, *and the* *dimension of space n, such that whenever ν is a measure-valued solution of the transport equation* (1.2) *and the initial value ν_0 has finite nth and $(n+1)$st moments, we have*

$$\fint_0^T E(t)^{\lambda r} L(t)^{-(1-\lambda)r}\, dt \geq C_2 \fint_0^T (t^{-1/3})^r\, dt \quad \text{for } T \geq C_3 L(0)^3, \tag{5.3}$$

where $E(t)$ and $L(t)$ are defined by (2.8) *and* (2.9)*, respectively, and the mean field $\theta(t)$ is defined by* (2.6)*.*

Taking $r = p$ and $\lambda = 1$ for $1 < p < 3$ in Theorem 5.1 gives Theorem 2.1.

THEOREM 5.2 (interface-reaction–controlled growth). *Let $n \geq 2$ be an integer. For any $0 \leq \lambda \leq 1$ and $0 < r < 3$ satisfying $\lambda r > 1$ and $(1-\lambda)r < 2$, there exist positive constants D_2 and D_3, depending only on λ, r, and the dimension of space n, such that whenever ν is a measure-valued solution of the transport equation* (1.8) *and the initial value ν_0 has finite nth and $(n+1)$st moments, we have*

$$\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geq D_2 \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T \geq D_3 L(0)^2, \tag{5.4}$$

where $E(t)$ and $L(t)$ are defined by (2.8) *and* (2.9)*, respectively, and the mean field $\theta(t)$ is defined by* (2.7)*.*

Taking $r = p+1$ and $\lambda = p/(p+1)$ for $1 < p < 2$ in Theorem 5.2 gives Theorem 2.2.

The remaining part of this section is devoted to proving the theorems above. To do this, we will need a change of variables as is done in [13]. In that paper, rather than directly working on distributions of particle radii R, Niethammer and Pego change the problems into equivalent ones expressed in terms of rescaled particle volumes $x(:= R^n)$ and work with a size-ranking function for particle volumes.

According to (1.1) and (1.7), the particle volume x satisfies the following growth law:

$$\dot{x} = a(x)\theta - b(x), \tag{5.5}$$

where

$$a(x) = \frac{nx^{1-1/n}}{x^{1/n} + \beta}, \quad b(x) = \frac{nx^{1-2/n}}{x^{1/n} + \beta} \quad \text{for volume-diffusion–controlled case}, \tag{5.6}$$

$$a(x) = nx^{1-1/n}, \quad b(x) = nx^{1-2/n} \quad \text{for interface-reaction–controlled case}, \tag{5.7}$$

and $\theta(t) = \int b(x)\, d\nu_t(x) / \int a(x)\, d\nu_t(x)$. Here ν is the measure-valued solution in the sense of distributions for the transport equation

$$\partial_t u + \partial_x\big((a(x)\theta - b(x))u\big) = 0. \tag{5.8}$$

The results of Niethammer and Pego are established by a further change of variables ([13]; see also [12]). For any size distribution of particles which is a probability measure μ on $[0, \infty)$, they define a *size-ranking function* $x = \hat{x}(\mu) : (0, 1] \to [0, \infty)$ by

$$x(\varphi) = \begin{cases} \sup\{y \mid \mu([y, \infty)) > \varphi\} & \text{for} \quad 0 < \varphi < 1, \\ 0 & \text{for} \quad \varphi = 1. \end{cases} \tag{5.9}$$

This is the right-continuous inverse of the tail distribution function $\varphi(x) = \mu([x,\infty))$. The map $\hat{x}$ gives a 1–1 correspondence between the set of Borel probability measures on $[0,\infty)$ and the set of right-continuous decreasing functions x on $(0,1]$ with $x(1) = 0$.

The following space for size ranking is introduced in [13]:

$$L^1_d = \{x : (0,1] \to \boldsymbol{R} |\ x \in L^1((0,1)),\ x(1) = 0,\ \text{and } x \text{ is decreasing} \\ \text{and right continuous on } (0,1]\}.$$

It is a closed subspace of $L^1((0,1))$. We will also perform our estimates in this space.

By statement 2.5.18(3) in [5], for any continuous function $f : (0,\infty) \to \mathbf{R}$ with compact support,

$$\int_0^1 f(x(\varphi))\, d\varphi = \int_0^\infty f(y)\, d\mu(y). \tag{5.10}$$

For any positive number $\alpha > 0$, $y \mapsto y^\alpha$ can be approximated by a monotonically increasing sequence of such functions, and thus by the monotone convergence theorem

$$\int_0^1 x(\varphi)^\alpha\, d\varphi = \int_0^\infty y^\alpha\, d\mu(y), \tag{5.11}$$

where both sides may be infinite. Hence $\mu \in \mathcal{P}_\alpha$ (Borel probability measures with finite αth moment) if and only if x is right-continuous decreasing on $(0,1]$ with $x(1) = 0$ and $\int_0^1 x(\varphi)^\alpha\, d\varphi < \infty$.

The growth law (5.5) can be rewritten as an integral equation,

$$x(t,\varphi) = x(0,\varphi) + \int_0^t (a(x(s,\varphi))\theta(s) - b(x(s,\varphi)))\, ds \tag{5.12}$$

with

$$\theta(t) = \int_0^{\bar{\varphi}(t)} b(x(t,\varphi))\, d\varphi \Big/ \int_0^1 a(x(t,\varphi))\, d\varphi \quad \text{for a.e. } t > 0, \tag{5.13}$$

where $\bar{\varphi}(t) := \sup\{\varphi | x(t,\varphi) > 0\}$.

Theorem 2.3 of [13] established the existence and uniqueness of the initial value problem for (5.12) and (5.13) under some assumptions ((H1)–(H5) in [13]) which our problems satisfy except for the 2D volume-diffusion–controlled growth model with $\beta = 0$. This theorem claims that for any $x_0 \in L^1_d$, there exists a unique function $x \in C([0,\infty), L^1_d)$ such that (5.12) and (5.13) hold with $x(0,\varphi) = x_0(\varphi)$. This is equivalent to the existence and uniqueness (Theorem 2.1 of [13]) of a weak-star continuous solution $\nu : [0,\infty) \to \mathcal{P}_1$ for the transport equation (5.8) in the sense of distributions on $(0,\infty) \times (0,\infty)$ with initial value $\nu_0 = \hat{x}^{-1}(x_0)$.

Proposition 6.1 of [13] established an L^1 compactness result for (5.12) and (5.13); namely, given $T \in (0,\infty)$, for a compact sequence of initial values $\{x_{0k}\} \subset L^1_d$, the corresponding sequence of solutions x_k is compact in $C([0,T], L^1_d)$ and any limit x is again a solution of (5.12) and (5.13).

Based on this result, in the appendix we prove an L^p compactness result for (5.12) and (5.13) for any $1 < p < \infty$; namely, given $T \in (0,\infty)$, for a sequence of initial values $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ which is compact in $L^p((0,1))$, the corresponding sequence of solutions x_k is compact in $C([0,T], L^p((0,1)))$ and any limit x is again a solution of (5.12) and (5.13).

Given $x_0 \in L^1_d \cap L^{(n+1)/n}((0,1))$, for any positive integer N, we divide the interval $(0,1)$ uniformly into N subintervals and define a function $x_{0N}(\varphi)$ by

$$x_{0N}(\varphi) = N \int_{(i-1)/N}^{i/N} x_0(\psi)\, d\psi \ (=: x^i_{0N}), \qquad \frac{i-1}{N} \le \varphi < \frac{i}{N},\ (i = 1, \dots, N). \tag{5.14}$$

Then $x_{0N} \in L^1_d \cap L^{(n+1)/n}((0,1))$ is piecewise constant, and $x_{0N} \to x_0$ in $L^{(n+1)/n}((0,1))$ as $N \to \infty$.

By the above compactness and uniqueness results, the solutions $\{x_N\}$ for (5.12) and (5.13) with initial values $\{x_{0N}\}$ converge in the space $C([0,T], L^{(n+1)/n}((0,1)))$ to the solution x for (5.12) and (5.13) with initial value x_0.

For any N, $\{x^i_{0N}\}_{i=1}^N$ gives a discrete collection of particles and the corresponding collection of radii $\{R_i := (x^i_{0N})^{1/n}\}$ undergoes coarsening determined by (1.1) or (1.7). Hence the estimates (3.16) and (4.39) claimed in Theorems 3.3 and 4.3 hold for

$$E_N(t) = \frac{\sum R_i(t)^{n-1}}{\sum R_i(t)^n} = \int_0^1 x_N(t,\varphi)^{(n-1)/n}\, d\varphi \Big/ \int_0^1 x_N(t,\varphi)\, d\varphi$$

and

$$L_N(t) = \frac{\sum R_i(t)^{n+1}}{\sum R_i(t)^n} = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\, d\varphi \Big/ \int_0^1 x_N(t,\varphi)\, d\varphi.$$

We will establish the convergence results for $E_N(t)$ and $L_N(t)$ in Lemma 5.4. To do this, let us first prove a general convergence result for L^p functions.

LEMMA 5.3. *For nonnegative functions $f_k, f \in L^p(\Omega)$ $(k = 1, 2, \dots)$ with $1 < p < \infty$ and Ω a bounded open subset of $\boldsymbol{R}^n$, if*

$$\int_\Omega |f_k(y)^p - f(y)^p|\, dy \to 0 \quad \text{as } k \to \infty, \tag{5.15}$$

then

$$\int_\Omega |f_k(y) - f(y)|^p\, dy \to 0 \quad \text{as } k \to \infty. \tag{5.16}$$

Proof. The convergence (5.15) implies that $\{f_k\}$ is bounded in $L^p(\Omega)$. Notice that $L^p(\Omega)$ is a reflexive Banach space since $1 < p < \infty$. Thus there exist a subsequence $\{f_{k_j}\}$ and $w \in L^p(\Omega)$ such that f_{k_j} converges weakly to w: $f_{k_j} \rightharpoonup w$ as $j \to \infty$. Hence

$$||w||_{L^p(\Omega)} \le \liminf_{j\to\infty} ||f_{k_j}||_{L^p(\Omega)} = ||f||_{L^p(\Omega)}. \tag{5.17}$$

By $f^p_{k_j} \to f^p$ in $L^1(\Omega)$, there exists a further subsequence, denoted the same, such that $f_{k_j}(y) \to f(y)$ for a.e. $y \in \Omega$. Hence, by Fatou's lemma and Hölder's inequality,

$$\int_\Omega f^p = \int_\Omega \liminf_{j\to\infty} f^{p-1} f_{k_j} \le \liminf_{j\to\infty} \int_\Omega f^{p-1} f_{k_j} = \int_\Omega f^{p-1} w \le \left(\int_\Omega f^p\right)^{1-\frac{1}{p}} \left(\int_\Omega w^p\right)^{1/p}.$$

Thus

$$||f||_{L^p(\Omega)} \le ||w||_{L^p(\Omega)}. \tag{5.18}$$

Comparing inequalities (5.17) and (5.18), we get $||f||_{L^p(\Omega)} = ||w||_{L^p(\Omega)}$. Thus

$$f_{k_j} \rightharpoonup w \quad \text{in} \;\; L^p(\Omega), \tag{5.19}$$

$$||f_{k_j}||_{L^p(\Omega)} \to ||w||_{L^p(\Omega)}. \tag{5.20}$$

Thus (see, e.g., [4])

$$f_{k_j} \to w \quad \text{in} \;\; L^p(\Omega), \tag{5.21}$$

and there exists a further subsequence of f_{k_j} that converges a.e. to w. Since $f_{k_j} \to f$ a.e. in Ω, we have $w = f$ and hence

$$f_{k_j} \to f \quad \text{in} \;\; L^p(\Omega). \tag{5.22}$$

The above argument works for every weakly convergent subsequence and hence the whole sequence f_k converges strongly to f in $L^p(\Omega)$. ∎

LEMMA 5.4. *For any $t > 0$, we have*

$$E_N(t) \to E(t) := \int_0^1 x(t,\varphi)^{(n-1)/n}\, d\varphi \Big/ \int_0^1 x(t,\varphi)\, d\varphi \quad \text{as } N \to \infty, \tag{5.23}$$

$$L_N(t) \to L(t) := \int_0^1 x(t,\varphi)^{(n+1)/n}\, d\varphi \Big/ \int_0^1 x(t,\varphi)\, d\varphi \quad \text{as } N \to \infty. \tag{5.24}$$

Proof. Fix $t > 0$. By the conservation of total mass and the convergence of initial value $x_{0N} \to x_0$ in $L^1((0,1))$,

$$\int_0^1 x_N(t,\varphi)\, d\varphi = \int_0^1 x_{0N}(\varphi)\, d\varphi \to \int_0^1 x_0(\varphi)\, d\varphi = \int_0^1 x(t,\varphi)\, d\varphi \tag{5.25}$$

as $N \to \infty$. By the compactness of $\{x_N\}$ in $C\big([0,T], L^p((0,1))\big)$ for all $T > 0$ and all $p > 1$,

$$\int_0^1 x_N(t,\varphi)^{(n+1)/n}\, d\varphi \to \int_0^1 x(t,\varphi)^{(n+1)/n}\, d\varphi \quad \text{as } N \to \infty. \tag{5.26}$$

The convergence of $L_N(t)$ to $L(t)$ is an immediate consequence of (5.25) and (5.26).

Define $f_N = x_N(t,\varphi)^{(n-1)/n}$, $f = x(t,\varphi)^{(n-1)/n}$, and $p = n/(n-1)$. Equation (5.25) implies that $f_N^p \to f^p$ as $N \to \infty$. Thus Lemma 5.3 implies $f_N \to f$ in $L^p((0,1))$ and consequently $f_N \to f$ in $L^1((0,1))$. Hence

$$\int_0^1 x_N(t,\varphi)^{(n-1)/n}\, d\varphi \to \int_0^1 x(t,\varphi)^{(n-1)/n}\, d\varphi. \tag{5.27}$$

The convergence of $E_N(t)$ to $E(t)$ is an immediate consequence of (5.25) and (5.27). ∎

To enable us to take limit in the estimates (3.16) and (4.39) claimed in Theorems 3.3 and 4.3, we will prove the following boundedness lemma for $E_N(t)$ and $L_N(t)$ and then apply Lebesgue's dominated convergence theorem.

LEMMA 5.5. *Given $T > 0$, there exist positive constants M_1, m_2, and M_2, depending only on n and T, such that*

$$0 < E_N(t) \le M_1, \quad m_2 < L_N(t) < M_2$$

uniformly in N and $0 \le t \le T$, with M_1, m_2, and M_2 positive constants depending only on n and T.

Proof. By (5.25), there exist positive constants $\hat{m}_1$ and $\hat{M}_1$ such that for all N and all $t \ge 0$,

$$\hat{m}_1 \le \int_0^1 x_N(t,\varphi)\,d\varphi \le \hat{M}_1, \quad \hat{m}_1 \le \int_0^1 x(t,\varphi)\,d\varphi \le \hat{M}_1. \tag{5.28}$$

Then by Hölder's inequality,

$$\int_0^1 x_N(t,\varphi)^{(n-1)/n}\,d\varphi \le \left(\int_0^1 x_N(t,\varphi)\,d\varphi\right)^{(n-1)/n} \le \hat{M}_1^{(n-1)/n}. \tag{5.29}$$

Hence

$$E_N(t) = \int_0^1 x_N(t,\varphi)^{(n-1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \le \hat{M}_1^{(n-1)/n}/\hat{m}_1 =: M_1. \tag{5.30}$$

By Hölder's inequality,

$$\hat{m}_1 \le \int_0^1 x_N(t,\varphi)\,d\varphi \le \left\{\int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi\right\}^{n/(n+1)}. \tag{5.31}$$

Thus

$$L_N(t) = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \ge \hat{m}_1^{(n+1)/n}/\hat{M}_1 =: m_2. \tag{5.32}$$

In the appendix, we will prove that there exists a positive increasing function $G(t)$ such that $\int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \le G(t) \le G(T)$. Thus, for all $0 \le t \le T$,

$$L_N(t) = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \le G(T)/\hat{m}_1 =: M_2. \tag{5.33}$$ ∎

The above boundedness results and Lebesgue's dominated convergence theorem guarantee that we can take limit as $N \to \infty$ in the estimates for coarsening rates for discrete systems (Theorems 3.3 and 4.3). This procedure gives us the estimates in Theorems 5.1 and 5.2, with E and L defined as in Lemma 5.4, for the coarsening rates for solutions of (5.12)+(5.13) with initial value $x_0 \in L^1_d \cap L^{(n+1)/n}((0,1))$.

Our ultimate goal is to get estimates for coarsening rates for measure-valued solutions of the transport equations (1.2) and (1.8), respectively. To do this, we will establish the 1–1 correspondence between these measure-valued solutions, which are distributions of particle radii, and volume size-ranking solutions for (5.12)+(5.13). The estimates for coarsening rates for these measure-valued solutions are immediate consequences of this 1–1 correspondence and the estimates for these size-ranking solutions.

For any initial particle radius distribution $\mu(R) \in \mathcal{P}_{n+1}$, we define a particle volume distribution $\hat{\mu}(x) = (T\mu)(x)$ by requiring

$$\int_0^\infty f(x)\,d\hat{\mu}(x) = \int_0^\infty f(R^n)\,d\mu(R) \tag{5.34}$$

for all continuous functions f with compact support. Then $\hat{\mu} \in \mathcal{P}_{(n+1)/n}$.

The size-ranking function $x_0(\varphi) = \hat{x}(\hat{\mu})$ defined as in (5.9) belongs to $L^1_d \cap L^{(n+1)/n}((0,1))$. Hence the solution $x(t,\varphi)$ of problem (5.12)+(5.13) with $x(0,\cdot) = x_0(\cdot)$ belongs to L^1_d and we can get the estimates as in Theorems 5.1 and 5.2 by the procedure described above.

It is proved in [13] that the mapping (5.9) is invertible and that under the assumptions (H1)–(H5), the weak-star continuous mapping $\hat{\nu} : [0,\infty) \to \mathcal{P}_1$ related with $x(t,\varphi)$ through (5.9) is the unique measure-valued solution of the transport equation (5.8) in the sense of distributions with initial value $\hat{\mu}$. For any $t \in [0,\infty)$, we define a Borel measure ν_t by requiring

$$\int_0^\infty f(R)\, d\nu_t(R) = \int_0^\infty f(x^{1/n})\, d\hat{\nu}_t(x) \tag{5.35}$$

for all continuous functions f with compact support. Then $\nu : [0,\infty) \to \mathcal{P}_n$ is weak-star continuous and is a measure-valued solution of the transport equation (1.2) or (1.8), with initial value $\nu_0 = \mu$. Again, since we can approximate a power function $y \mapsto y^\alpha (\alpha > 0)$ by a monotonically increasing sequence of continuous functions with compact support, we get the following moment equivalence identity for ν and $\hat{\nu}$:

$$\int_0^\infty R^\alpha\, d\nu_t(R) = \int_0^\infty x^{\alpha/n}\, d\hat{\nu}_t(x) \quad \text{for any } \alpha > 0. \tag{5.36}$$

On the other hand, if we have a measure-valued solution $\nu : [0,\infty) \to \mathcal{P}_n$ for (1.2) or (1.8) with a given initial value μ, we can define $\hat{\nu} : [0,\infty) \to \mathcal{P}_1$ by (5.35) and $\hat{\nu}$ will be a measure-valued solution for (5.8) with initial value $\hat{\mu}$ defined by (5.34). The uniqueness of $\hat{\nu}$ then implies the uniqueness of ν.

The above analysis, together with the moment equivalence statements (5.11) and (5.36), gives us Theorems 5.1 and 5.2 on coarsening rates for mean-field models with general initial distributions of particle radii.

Appendix. In this appendix, we will establish a compactness result for solutions $x(t,\varphi)$ of problem (5.12)+(5.13) with $x(0,\cdot) = x_0(\cdot)$ for any $x_0 \in L^1_d \cap L^p((0,1))$ with $1 < p < \infty$ under the same assumptions (H1)–(H4) as in [13]. Note that the two models we considered fall into this category except for the case $n = 2, \beta = 0$ of the volume-diffusion–controlled growth model (and this is the reason why we do not include this case in our estimates for coarsening rates with general size distribution).

PROPOSITION A.1. *Fix $T \in (0,\infty)$ and consider a sequence $\{x_k\}_{k=1}^\infty$ of solutions to* (5.12)+(5.13) *for $0 \le t \le T$ with initial values $x_k(0,\varphi) = x_{0k}(\varphi)$ $(\varphi \in (0,1))$. Assume that the sequence of initial data $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ is compact in $L^p((0,1))$ for some $1 < p < \infty$ with $c_1 := \inf_k \int_0^1 x_{0k} > 0$. Then $\{x_k\}$ is compact in $C([0,T], L^p((0,1)))$ and any limit x is again a solution of* (5.12)+(5.13).

Proof. By Hölder's inequality, the assumption that $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ is compact in $L^p((0,1))$ implies that $\{x_{0k}\}$ is compact in $L^1((0,1))$. Hence, by Proposition 6.1 in [13], $\{x_k\}$ is compact in $C([0,T], L^1_d)$ and any limit x is again a solution of (5.12)+(5.13). We will follow the strategy of the proof of Lemma 6.2 in [13] to prove that x_k is compact in $L^p((0,1))$.

It has been shown in [13] that $\theta(t)$ is uniformly bounded on [0,T]. The assumptions $(H1)$–$(H4)$ together with the boundedness of θ imply that there exists a positive constant C, depending only on T, such that

$$|a(x)\theta(t) - b(x)| \le C(1 + x)$$

for all $x \geq 0$. By the generalized Arzelà–Ascoli theorem, to show that $\{x_k\}$ is compact in $C([0,T], L^p((0,1)))$, we need to prove the following three steps:

(1) uniform boundedness of $\{\int_0^1 x_k^p(t,\varphi)\,d\varphi\}$ for all $t \in [0,T]$ and all k,

(2) for fixed $t \in (0,T)$, $\{x_k(t,\cdot)\}$ is compact in $L^p((0,1))$,

(3) $\sup_k ||x_k(t_1,\cdot) - x_k(t_2,\cdot)||_{L^p((0,1))} \to 0$ as $|t_1 - t_2| \to 0$.

To show (1), define $F_\delta(t) = \int_\delta^1 x_k^p(t,\varphi)\,d\varphi$ for $\delta > 0$. Then $F_\delta < \infty$ since $x_k(t,\cdot)$ is decreasing and

$$\begin{aligned} F_\delta(t) &= \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + \int_\delta^1 \int_0^t p x_k^{p-1} \partial_s x_k(s,\varphi)\,ds\,d\varphi \\ &= \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + \int_\delta^1 \int_0^t p x_k^{p-1} (a(x_k)\theta - b(x_k))\,ds\,d\varphi \\ &\leq \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + Cp \int_\delta^1 \int_0^t x_k^{p-1}(1 + x_k)\,dt\,d\varphi. \end{aligned}$$

By Young's inequality,

$$x_k^{p-1} \leq \frac{p-1}{p} x_k^p + \frac{1}{p}. \tag{A.1}$$

Thus

$$\begin{aligned} F_\delta(t) &\leq \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + C \int_\delta^1 \int_0^t ((2p-1)x_k^p + 1)\,dt\,d\varphi \\ &\leq \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + CT + C(2p-1) \int_0^t \int_\delta^1 x_k^p\,d\varphi\,dt. \end{aligned}$$

By Gronwall's inequality,

$$F_\delta(t) \leq \int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + CT + C(2p-1)e^{C(2p-1)t} \left(\int_\delta^1 x_{0k}^p(\varphi)\,d\varphi + CT \right). \tag{A.2}$$

The compactness of x_{0k} in $L^p((0,1))$ implies that there exists positive constant C_1 such that $\int_0^1 x_{0k}^p(\varphi)\,d\varphi \leq C_1$ for all k. Thus, by taking $\delta \to 0$ in (A.2) we get

$$\int_0^1 x_k^p(t,\varphi)\,d\varphi \leq C_1 + CT + C(2p-1)e^{C(2p-1)t}(C_1 + CT) =: G(t) \leq G(T). \tag{A.3}$$

Here G is an increasing function of t and does not depend on k. Hence (1) is proved.

It is shown in the proof of Lemma 6.2 in [13] that, for fixed t, there exists a pointwise convergent subsequence, still denoted as $\{x_k\}$ for simplicity. Therefore, to prove (2) we need only show that $\{x_k\}$ is equi-integrable. Since $x_k(t,\cdot)$ are decreasing, it is enough to show

$$\sup_k \int_0^\varepsilon x_k(t,\varphi)\,d\varphi \to 0 \quad \text{as } \varepsilon \to 0. \tag{A.4}$$

$$\begin{aligned} \int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi &= \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + \int_0^\varepsilon \int_0^t p x_k^{p-1}(s,\varphi) \partial_s x_k(s,\varphi)\,ds\,d\varphi \\ &= \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + \int_0^\varepsilon \int_0^t p x_k^{p-1}(s,\varphi)(a(x_k)\theta - b(x_k))\,ds\,d\varphi \end{aligned} \tag{A.5}$$

$$\leq \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + Cp\int_0^\varepsilon\int_0^t x_k^{p-1}(1+x_k)\,ds\,d\varphi$$
$$\leq \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + C\int_0^\varepsilon\int_0^t ((2p-1)x_k^p+1)\,ds\,d\varphi \quad \text{by (A.1)}$$
$$\leq \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon + C(2p-1)\int_0^t\int_0^\varepsilon x_k^p\,d\varphi\,ds.$$

By Gronwall's inequality,

$$\text{(A.6)} \qquad \int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi \leq \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon + (2p-1)Ce^{(2p-1)Ct}\left(\int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon\right).$$

We can assume without loss of generality that $x_{0k} \to x_0$ in $L^p((0,1))$. Hence $\sup_k \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi \to 0$ as $\varepsilon \to 0$. By (A.6), $\sup_k \int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi \to 0$ as $\varepsilon \to 0$ and (2) is proved.

Now let us prove (3). Assume $t_1 < t_2$.

$$\text{(A.7)} \quad \int_0^1 |x_k^p(t_1,\varphi) - x_k^p(t_2,\varphi)|\,d\varphi = p\int_0^1 \left|\int_{t_1}^{t_2} x_k^{p-1}\partial_t x_k(t,\varphi)\,dt\right| d\varphi$$
$$= p\int_0^1 \left|\int_{t_1}^{t_2} x_k^{p-1}(a(x_k)\theta(t) - b(x_k))\,dt\right| d\varphi$$
$$\leq Cp\int_0^1\int_{t_1}^{t_2} x_k^{p-1}(1+x_k)\,dt\,d\varphi$$
$$\leq C\int_0^1\int_{t_1}^{t_2} ((2p-1)x_k^p+1)\,dt\,d\varphi \quad \text{by (A.1)}$$
$$\leq C(2p-1)(G(T)+1)|t_2 - t_1| \quad \text{by (A.3)}.$$

Thus (3) is true and the proposition is proved. ∎

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