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MOTION OF INTERFACES GOVERNED BY THE CAHN–HILLIARD EQUATION WITH HIGHLY DISPARATE DIFFUSION MOBILITY*

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Abstract. We consider a two-phase system governed by a Cahn–Hilliard type equation with a highly disparate diffusion mobility. It has been observed from recent numerical simulations that the microstructure evolution described by such a system displays a coarsening rate different from that associated with the Cahn–Hilliard equation having either a constant diffusion mobility or a mobility that degenerates in both phases. Using the asymptotic matching method, we derive sharp interface models of the system under consideration to theoretically analyze the interfacial motion with respect to different scales of time t. In a very short time regime, the transition layer stabilizes into the well-known hyperbolic tangent single-layer profile. On an intermediate $t = O(1)$ time regime, due to the small mobility in one of the phases, the sharp interface limit is a one-sided Stefan problem, determined by data in the phase with constant nonzero mobility. On a slower $t = O(\varepsilon^{-1})$ time scale, the leading order dynamics is a one-sided Hele–Shaw problem. When this one-sided Hele–Shaw dynamics is equilibrated, the system evolves in $t = O(\varepsilon^{-2})$ time scale according to the combination of a one-sided modified Mullins–Sekerka problem in the phase with nonzero constant mobility and a nonlinear diffusion process that solves a quasi-stationary porous medium equation in the phase with small mobility. Scaling arguments suggest that there should be a crossover in the coarsening rate from $t^{1/3}$ to $t^{1/4}$.

Key words. Cahn–Hilliard equation, asymptotic analysis, coarsening, motion of interfaces

AMS subject classifications. 35B40, 35Q92, 74N20, 82C26

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1. Introduction.

Cahn–Hilliard equations are well-known diffuse interface models that describe the dynamics of a two-phase system undergoing spinodal decomposition. A general form can be written as

$$u_t = \nabla \cdot (M(u)\nabla\mu) \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

$$\mu := -\varepsilon^2 \Delta u + F'(u). \tag{1.2}$$

Here, $M = M(u)$ denotes the diffusion mobility, μ is the variational derivative of the free energy

$$E(u) = \int_\Omega \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right\} dx \tag{1.3}$$

with respect to the order parameter u which represents the relative concentration difference of the two phases, and $F = F(u)$ is a double-well potential with two equal minima at $u^\pm = \pm 1$ corresponding to the two phases. A widely used choice of $F = F(u)$ is

$$F(u) = \frac{(u^2 - 1)^2}{4}. \tag{1.4}$$

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Since in general the two phases in the system—for example, the two ingredients in binary alloys, or water and polymer in a water-polymer mixture—have very different physical or chemical properties, it is natural that the values of the diffusion mobility M depend on the phases. And it turns out that the properties of the Cahn–Hilliard equation depend crucially on M.

One feature of Cahn–Hilliard equations is that the underlying system can be used to describe a coarsening phenomenon which refers to an increase of the typical length scales in the spatial structures due to the reduction of the interfacial energy. How such a coarsening process affects the mechanical properties of a material has been a subject of interest in materials science and engineering for a long time. For the Cahn–Hilliard equation, it is often observed that some characteristic length scale l increases as a power of time, and such power law depends on the diffusion mobility $M = M(u)$.

When the diffusion mobility is a constant independent of the phases, the migration of the sharp interface separating the two phases was studied by Pego [13] using formal asymptotic analysis. Let Γ be the interface separating the two phases. Mathematically we take Γ as the zero level set of the order parameter u. Letting $k_i, i = 1, \ldots, n-1$, be the principal curvatures of Γ and expanding them in powers of ε, we have

$$k_i = k_{i0} + \varepsilon k_{i1} + \varepsilon^2 k_{i2} + \cdots \tag{1.5}$$

and the mean curvature $\kappa := \sum_{i=1}^{n-1} k_i$ accordingly has an expansion in powers of ε of the form $\kappa = \kappa_0 + \varepsilon\kappa_{01} + O(\varepsilon^2)$. Let $\Omega_\pm$ be the regions occupied by the two phases corresponding to $u = \pm 1$, respectively. It is shown that in the time scale $t_1 = \varepsilon t$, the sharp interface limit of the constant mobility Cahn–Hilliard equation is the Mullins–Sekerka model

$$\Delta\mu_1 = 0 \quad \text{in } \Omega \setminus \Gamma, \tag{1.6}$$

$$\mu_1 = -\frac{S}{[U]}\kappa_0 \quad \text{on } \Gamma, \tag{1.7}$$

$$V = -\frac{1}{[U]}[\partial_{\mathbf{n}}\mu_1]_-^+ \quad \text{on } \Gamma. \tag{1.8}$$

Here $\kappa_0 := k_{10} + \cdots + k_{n-1,0}$ is the leading order of the mean curvature of Γ. The orientation of Γ is so that $\kappa_0 > 0$ when the center of curvature lies in Ω_-. V is the normal velocity of Γ, positive when Γ migrates toward Ω_+. The jump of flux $[\partial_{\mathbf{n}}\mu_1]_-^+ = \mathbf{n}\cdot(\nabla\mu_1^+ - \nabla\mu_1^-)$, where $\mathbf{n}$ is the normal pointing toward Ω_+. The constants

$$S = \int_{-\infty}^{\infty} U'(z)^2\, dz, \quad [U] = \int_{-\infty}^{\infty} U'(z)\, dz \tag{1.9}$$

with

$$U(z) = \tanh\frac{z}{\sqrt{2}} \tag{1.10}$$

being the transition-layer profile.

The Mullins–Sekerka model possesses a property of scaling invariance (see, e.g., [4, 5, 14]). Taking M_1, X, T_1 as typical scales for the chemical potential, length, and time, respectively, we rescale μ_1, x, and t_1 so that

$$\mu_1 = M_1\hat{\mu}_1, \quad x = X\hat{x}, \quad t_1 = T_1\hat{t}_1. \tag{1.11}$$

Then

$$\kappa_0 = \frac{1}{X}\hat{\kappa}_0, \quad V = \frac{X}{T_1}\hat{V}, \quad [\partial_{\mathbf{n}}\mu_1]_-^+ = \frac{M_1}{X}[\partial_{\mathbf{n}}\hat{\mu}_1]_-^+ \tag{1.12}$$

and (1.6)–(1.8) becomes

$$\Delta\hat{\mu}_1 = 0 \quad \text{in } \Omega \setminus \Gamma, \tag{1.13}$$

$$M_1\hat{\mu}_1 = -\frac{1}{X}\frac{S}{[U]}\hat{\kappa}_0 \quad \text{on } \Gamma, \tag{1.14}$$

$$\frac{X}{T_1}\hat{V} = -\frac{M_1}{X}\frac{1}{[U]}[\partial_{\mathbf{n}}\hat{\mu}_1]_-^+ \quad \text{on } \Gamma. \tag{1.15}$$

If we take $M_1 = X^{-1}$ and $T_1 = X^3$, then (1.13)–(1.15) has exactly the same form as (1.6)–(1.8). This scaling invariance indicates that the Mullins–Sekerka model generically exhibits a spatial-temporal power law relation $X \sim c_1 T_1^{1/3}$. (For the relation between scaling invariance and spatial-temporal power law relation, see, e.g., [14].) Consequently in the time regime $t_1 = \varepsilon t$, the constant mobility Cahn–Hilliard equation exhibits a coarsening rate $l \sim c_1 t_1^{1/3}$, where l is a characteristic length scale. Such a $1/3$ power law was observed in experiments and numerical simulations [7, 11, 15], although discrepancies were also observed. For instance, based on two-dimensional numerical simulations, [8] reported a very sensitive dependence of the coarsening rate on the type of data with transient exponents ranging from $1/6$ to $1/3$.

The case where M is degenerate in both phases, e.g., $M(u) = 1 - u^2$, was studied by Cahn, Elliott, and Novick-Cohen [3] for a double-well potential

$$\hat{F}(u) = \frac{\theta}{2}\Big((1+u)\ln(1+u) + (1-u)\ln(1-u)\Big) + \frac{1}{2}(1-u^2). \tag{1.16}$$

Apparently $\hat{F}$ is different from (1.4) as $\hat{F}$ is not smooth at $u = \pm 1$. $\hat{F}$ has two equal minima at $\pm\beta(\theta)$, where $\beta(\theta) = 1 - \hat{\mathcal{R}}(\theta)$ and $\hat{\mathcal{R}}$ is a small term depending on θ.

Under some assumptions on the behavior of the solution and on θ, they found that in the time scale $t_2 = \varepsilon^2 t$, the sharp interface limit of the degenerate mobility Cahn–Hilliard equation is the surface diffusion equation

$$V = -c\Delta_s \kappa_0, \tag{1.17}$$

where c is a constant and Δ_s is the Laplace–Beltrami operator on Γ. Again, scaling arguments indicate a coarsening rate of $l \sim c_2 t_2^{1/4}$ for the degenerate mobility Cahn–Hilliard equation. Such a rate has also been observed in numerical simulations. Similar computational findings on the degenerate mobility Cahn–Hilliard equation with double-well potential of the form (1.4) can also be found in [17] and the references cited therein. In [2], it is suggested by heuristic argument that, at least for the smooth double well potential F in (1.4), even though the mobility is degenerate in both phases, there is a weaker diffusion process—which is called subdiffusion in [2]—in both phases that contributes to the coarsening process with a $1/4$ power law.

In [16], numerical simulations were performed for the case that the mobility is highly disparate, that is, $M(u) = O(1)$ in Ω_+ while it is very small in Ω_-. To analyze the coarsening kinetics, several approaches are used in [16] to determine the average length scale as a function of time t, including the temporal structure function, the pair correlation function, and the direct counting of particle sizes in real space. Based on these computational results, it is observed that in this situation, if a power law is assumed, data fitting suggests that the coarsening rate is $l \sim t^{1/3.3}$ at the time scale that the microstructure can be numerically resolved. This numerically estimated power is quite unusual. It is the objective of this paper to conduct some theoretical analysis of this observation. Our analysis not only gives a good explanation to the claims made in [16] but also offers new insight into the dynamics of microstructure evolution in two-phase systems. Since $u = \pm 1 + O(\varepsilon)$ in $\Omega_\pm$, for simplicity we take

$$M(u) = 1 + u = \begin{cases} 2 + O(\varepsilon) & \text{in } \Omega_+, \\ O(\varepsilon) & \text{in } \Omega_-. \end{cases} \tag{1.18}$$

By studying the motion of the interface using asymptotic analysis, we show that if a power law relation is expected, then in the time scale $t_1 = \varepsilon t$, it is possible for a relation $l \sim C_1 t_1^{1/3}$ to hold in an early period.

Afterward, because of the degeneracy of the diffusion mobility, if some regions that are subsets of Ω_+ are disjoint from other regions of Ω_+, the evolution of these regions will be isolated from the rest of the system, causing the t_1 dynamics to slow down and equilibrate, while other nonisolated regions of Ω_+ still evolve under nontrivial t_1 dynamics.

One interesting case is when Ω_+ is a minor phase occupying a small fraction of the system. In this case Ω_+ breaks into a collection of disjoint components of various sizes. After the t_1 evolution of each component is equilibrated, the system coarsens in the $t_2 = \varepsilon^2 t$ time scale with a coarsening rate $l \sim C_2 t_2^{1/4}$.

So in the coarsening dynamics, there is in fact a transient behavior. But because of the complexity of the system, it is very difficult to numerically observe this transition. So the unusual power law in [16] may be attributed to the fact that in general we may not expect a simple power law relation in the time regime in which the numerical simulations were performed.

2. Main results. In this section we summarize the main results of this paper.

Using asymptotic matching methods we study the motion of the interface in early, intermediate, and late time scales. We find that the leading order of the transition layer is equilibrated on the time scale $T_2 = t/\varepsilon^2$ and $T_1 = t/\varepsilon$ into the well-known profile given in (1.10). During this process, the leading order of the initial value away from the interface is preserved.

On the time scale $t = O(1)$, the evolution of the interface is determined by a one-sided Stefan problem,

$$\partial_t u_0 = \nabla \cdot \big((1 + u_0)\nabla \mu_0\big), \quad \mu_0 = F'(u_0) \quad \text{in } \Omega_+, \tag{2.1}$$

$$u_0^\pm = \pm 1 \quad \text{on } \Gamma, \tag{2.2}$$

$$V = -\partial_{\mathbf{n}} \mu_0^+. \tag{2.3}$$

We call this a one-sided problem because the normal velocity of the interface is determined solely by data in Ω_+.

In a later time scale $t_1 = \varepsilon t$, the interfacial motion follows a one-sided Hele–Shaw problem,

$$\Delta \mu_1 = 0 \quad \text{in } \Omega_+, \tag{2.4}$$

$$\mu_1 = -\frac{S}{[U]}\kappa_0 \quad \text{on } \Gamma, \tag{2.5}$$

$$V = -\partial_{\mathbf{n}} \mu_1^+ \quad \text{on } \Gamma, \tag{2.6}$$

where the constants S and $[U]$ are given in (1.9). The contribution from data in Ω_-

is an $O(\varepsilon)$ correction of the normal velocity, which also includes a surface diffusion process. We do not analyze the details of this correction because of the lack of an explicit expression.

Note that in [10], Glasner derived a diffuse-interface model for the one-phase Hele–Shaw flow. His model is very similar to our Cahn–Hilliard equation with highly disparate diffuse mobility. Our derivation of (2.4)–(2.6) is also consistent with that given in [10].

As for the coarsening rate, we need a careful scaling argument. In the t_1 time scale, notice that (2.4)–(2.6) is invariant under the scaling (1.11) with $M_1 = X^{-1}$ and $T_1 = X^3$, similar to the Mullins–Sekerka model (1.6)–(1.8), so we may expect a length-time relation $l \sim C_1 t_1^{1/3}$, provided V remains a dominating $O(1)$ quantity.

However, the one-sided feature of (2.4)–(2.6) has a distinct effect. Generically, if Ω_+ breaks into several disjoint components, then each component is isolated from the others and evolves independently of the others, according to the one-sided Hele–Shaw problem (2.4)–(2.6), unless collisions of these components occur. This becomes apparent when Ω_+ is a small fraction of the total system.

Consequently, for some disjoint components of Ω_+, the evolution of their boundaries are "localized" and can slow down so that V becomes small, literally making the normal velocity an $O(\varepsilon)$ quantity. This can happen in a short period of time, especially if the disjoint components are close to spheres and also for numerical simulations when ε is in fact not too small.

Assume Ω_+ is a collection of disjoint components. When the $O(1)$ order dynamics in t_1 time scale equilibrates, we need to study the evolution of the interface in $t_2 = \varepsilon^2 t$ time scale. Since the leading order t_1-dynamics is equilibrated, the leading order mean curvature κ_0 of each component becomes a constant. Consequently the surface diffusion effect disappears as the leading order t_1-dynamics equilibrates.

It turns out that the evolution in the t_2 time scale is driven by a diffusion process in Ω_+ involving the first order correction of the mean curvature and a nonlinear diffusion process in Ω_- which solves a quasi-stationary porous medium equation. The normal velocity of the interface is determined by the fluxes of these two diffusion processes.

$$\Delta\mu_2 = 0 \quad \text{in } \Omega_+, \tag{2.7}$$

$$\mu_2 = -\kappa_0^2 \frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]} \quad \text{on } \Gamma, \tag{2.8}$$

$$\nabla\cdot(\mu_1\nabla\mu_1) = 0 \quad \text{in } \Omega_-, \tag{2.9}$$

$$\mu_1 = -\frac{S}{[U]}\kappa_0 \quad \text{on } \Gamma, \tag{2.10}$$

$$V = -\partial_{\mathbf{n}}\mu_2^+ + \frac{1}{4}\mu_1^-\partial_{\mathbf{n}}\mu_1^- \quad \text{on } \Gamma. \tag{2.11}$$

Here S_1 is a constant (see the later derivation for a precise definition) and κ_{01} is the $O(\varepsilon)$ correction term to the mean curvature of Γ. The modified Dirichlet boundary condition for μ_2 in (2.8), and the porous medium diffusion equation for μ_1 (2.9), induce a different scaling invariance of the system from that of (2.4)–(2.6), resulting in a different spatial-temporal relation.

To derive the spatial-temporal relation, we take M_1, M_2, X, T_2 as typical scales for the first and second order chemical potentials, length, and time so that

$$\mu_1 = M_1\hat{\mu}_1, \quad \mu_2 = M_2\hat{\mu}_2, \quad x = X\hat{x}, \quad t_2 = T_2\hat{t}_2. \tag{2.12}$$

Then

$$V = \frac{X}{T_2}\hat{V}, \quad \partial_{\mathbf{n}}\mu_1^- = \frac{M_1}{X}\partial_{\mathbf{n}}\mu_1^-, \quad \partial_{\mathbf{n}}\mu_2^+ = \frac{M_2}{X}\partial_{\mathbf{n}}\hat{\mu}_2^+, \quad \kappa = \frac{1}{X}\hat{\kappa}. \tag{2.13}$$

Since ε measures the thickness of the transition region, it should be rescaled according to $\varepsilon = X\hat{\varepsilon}$. So $\kappa = \frac{1}{X}\hat{\kappa}$ translates into

$$\kappa_0 + \varepsilon\kappa_{01} + \cdots = \frac{1}{X}\left(\hat{\kappa}_0 + \hat{\varepsilon}\hat{\kappa}_{01} + \cdots\right). \tag{2.14}$$

The balance of $O(1)$ and $O(\varepsilon)$ terms on both sides gives $\kappa_0 = \frac{1}{X}\hat{\kappa}_0$ and $\kappa_{01} = \frac{1}{X^2}\hat{\kappa}_{01}$. In the rescaled variables (2.7)–(2.11) translate into

$$\hat{\Delta}\hat{\mu}_2 = 0 \quad \text{in } \Omega_+, \tag{2.15}$$

$$M_2\hat{\mu}_2 = \frac{1}{X^2}\left(-\hat{\kappa}_0^2\frac{S_1}{[U]} - \hat{\kappa}_{01}\frac{S}{[U]}\right) \quad \text{on } \Gamma, \tag{2.16}$$

$$\hat{\nabla}\cdot(\hat{\mu}_1\hat{\nabla}\hat{\mu}_1) = 0 \quad \text{in } \Omega_-, \tag{2.17}$$

$$M_1\hat{\mu}_1 = \frac{1}{X}\hat{\kappa}_0 \quad \text{on } \Gamma, \tag{2.18}$$

$$\frac{X}{T_2}\hat{V} = -\frac{M_2}{X}\partial_{\mathbf{n}}\hat{\mu}_2^+ + \frac{M_1^2}{X}\hat{\mu}_1^-\partial_{\mathbf{n}}\mu_1^- \quad \text{on } \Gamma. \tag{2.19}$$

By (2.16) and (2.18), naturally M_2 should be chosen as X^{-2} and M_1 should be X^{-1}. If in addition we choose $T_2 = X^4$, then (2.15)–(2.19) is the same as (2.7)–(2.11), except for the hats on the unknowns. This scaling invariance under $T_2 = X^4$, or equivalently $X = T_2^{1/4}$, indicates a coarsening rate $l \sim C_2 t_2^{1/4} = C_2\varepsilon^{1/4}t_1^{1/4}$.

The above argument suggests that for the Cahn–Hilliard equation with one-sided degenerate mobility, the coarsening rate generically should have an upper bound $t_1^{1/3}$ and a lower bound $\varepsilon^{1/4}t_1^{1/4}$ and a single power law relation may not be expected. Thus an exponent with an intermediate value between 1/3 and 1/4 might be obtained when attempting to fit with a power law numerically as illustrated in [16].

In contrast, for the Cahn–Hilliard equation with a constant mobility, the interfacial dynamics is described by the Mullins–Sekerka model (1.6)–(1.8). Since it is determined by data on both sides of the interface, even if Ω_+ breaks into disjoint components, these components still interact through the diffusion field in Ω_-. As a result, the leading order dynamics remains dominating and the length-time relation remains a 1/3 power law.

As for the case when the diffusion mobility is degenerate in both phases, in the t_1 time scale, the velocity on the $O(1)$ order is 0. In the t_2 dynamics, there is a nontrivial surface diffusion process. Consequently the length-time relation is a 1/4 power law.

3. Moving inner local frame. To handle the inner expansion, we need to establish a local coordinate system in a neighborhood of the interface $\Gamma(t)$, which is defined as the zero level set of the order parameter u. This has been done by many authors in various situations under various assumptions; see, e.g., [3, 6, 9, 13]. Since our asymptotic analysis requires very precise expansions in several orders of ε, we describe the details for the convenience of readers.

We assume that for any t, $\Gamma(t)$ is a collection of simple, closed, smooth surfaces in $\mathbb{R}^n$. So we can find a parametric representation, at least locally,

$$\Gamma(t) = \{\phi(s,t) : s = (s_1, \ldots, s_{n-1}) \in Q(t) \subset \mathbb{R}^{n-1}\}. \tag{3.1}$$

To simplify the calculation, we choose the parametrization so that s_i is the arc length of the ith coordinate curve and the coordinate curves are lines of curvature. That is,

$$\mathbf{T}^i := \frac{\partial \phi}{\partial s_i}, \quad i = 1, \ldots, n-1, \tag{3.2}$$

form an orthonormal basis for the tangent space at point $\phi(s,t)$ and, letting $\mathbf{n}(s,t)$ be the outer normal vector of Γ pointing to Ω_+, we have

$$\frac{\partial \mathbf{T}^i}{\partial s_i} = -k_i \mathbf{n}, \quad \frac{\partial \mathbf{n}}{\partial s_i} = k_i \mathbf{T}^i, \quad i = 1, \ldots, n-1, \tag{3.3}$$

where k_i are principal curvatures. Here we choose the orientation of Γ so that if Γ is convex, then the center of curvature lies in Ω_- and the mean curvature is positive.

We assume further that for every point x in a neighborhood of Γ, there is a unique point $\phi(s,t)$ that is the orthogonal projection of x onto Γ. Then we can define a unique normal signed distance $\rho(x,t)$ from x to Γ,

$$\rho(x,t) := (x - \phi(s,t)) \cdot \mathbf{n}(s,t). \tag{3.4}$$

Now we have a transform of coordinates from (x,t) to (s,r,t) defined by

$$x = \phi(s,t) + r\mathbf{n}(s,t), \tag{3.5}$$

where $r = \rho(x,t)$.

In this paper we will use the Einstein summation convention, where repeated indices are summed over unless otherwise indicated. Componentwise, writing $\mathbf{n} = (N_1, \ldots, N_n)$ and $\mathbf{T}^i = (T_1^i, \ldots, T_n^i)$, (3.3) and (3.4) can be written as

$$\frac{\partial T_j^i}{\partial s_i} = -k_i N_j, \quad \frac{\partial N_j}{\partial s_i} = k_i T_j^i, \quad i = 1, \ldots, n-1, \ j = 1, \ldots, n, \tag{3.6}$$

$$\rho = (x_i - \phi_i(s,t)) N_i(s,t). \tag{3.7}$$

In the following lemma, we summarize some facts that will be useful for later calculations.

LEMMA 3.1.

(i) $-\dfrac{\partial \rho}{\partial t}$ *is the normal velocity of* Γ *at* $\phi(s,t)$.

(ii) $\nabla_x s_i = \dfrac{1}{1 + r k_i} \mathbf{T}^i, \quad \Delta_x s_i = -r \dfrac{\partial k_i}{\partial s_i} \dfrac{1}{(1 + r k_i)^3}, \quad i = 1, \ldots, n-1.$

(iii) $\nabla_x r = \mathbf{n}, \quad \Delta_x r = \displaystyle\sum_{j=1}^{n-1} \frac{k_j}{1 + r k_j}.$

(iiv) *In the local coordinates* (s, r), *the Cartesian Laplacian* Δ_x *has the following expression:*

$$\Delta_x = \sum_{j=1}^{n-1} \frac{1}{(1+rk_j)^2} \frac{\partial^2}{\partial s_j^2} + \left(\sum_{j=1}^{n-1} \frac{k_j}{1+rk_j} \right) \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \sum_{j=1}^{n-1} \left(-r \frac{\partial k_j}{\partial s_j} \frac{1}{(1+rk_j)^3} \right) \frac{\partial}{\partial s_j}. \tag{3.8}$$

Proof. (i) Since $x = \phi(s,t) + \rho(x,t)\mathbf{n}$, taking derivative in t we have

$$0 = \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{\partial \rho}{\partial t}\mathbf{n} + \rho \left(\frac{\partial \mathbf{n}}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \mathbf{n}}{\partial t} \right).$$

Since $\frac{\partial \phi}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \phi}{\partial t}$ is the velocity of the interface $\Gamma(t)$ and $\frac{\partial \mathbf{n}}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \mathbf{n}}{\partial t}$ is tangent to $\Gamma(t)$, $-\frac{\partial \rho}{\partial t}$ is the value of the normal velocity.

(ii) By (3.5), we have $x_i = \phi_i + rN_i$. Taking derivative in x_j, we have

$$\delta_{ij} = \frac{\partial \phi_i}{\partial s_m}\frac{\partial s_m}{\partial x_j} + \frac{\partial r}{\partial x_j}N_i + r\frac{\partial N_i}{\partial s_m}\frac{\partial s_m}{\partial x_j} = \sum_m T_i^m (1+rk_m)\frac{\partial s_m}{\partial x_j} + \frac{\partial r}{\partial x_j}N_i.$$

Multiplying both sides by T_i^m, summing over i, and using the facts that $T_i^m T_i^k = \delta_{mk}$, $T_i^m N_i = 0$, we get

$$T_j^m = (1+rk_m)\frac{\partial s_m}{\partial x_j}, \qquad m = 1, \ldots, n-1, \;\; j = 1, \ldots, n.$$

This gives $\nabla_x s_i = (1+rk_i)^{-1}\mathbf{T}^i$, $i = 1, \ldots, n-1$.

For any i, j, since $\mathbf{T}^i \cdot \mathbf{T}^j = \delta_{ij}$, we have

$$0 = \frac{\partial \mathbf{T}^i}{\partial s_j} \cdot \mathbf{T}^j + \mathbf{T}^i \cdot \frac{\partial \mathbf{T}^j}{\partial s_j} = \frac{\partial \mathbf{T}^i}{\partial s_j} \cdot \mathbf{T}^j + \mathbf{T}^i \cdot (-k_j)\mathbf{n} = \frac{\partial \mathbf{T}^i}{\partial s_j} \cdot \mathbf{T}^j. \tag{3.9}$$

So

$$\begin{aligned}
\Delta_x s_i = \nabla_x \cdot \left(\frac{1}{1+rk_i}\mathbf{T}^i \right) &= -\frac{k_i \nabla_x r + r\frac{\partial k_i}{\partial s_j}\nabla_x s_j}{(1+rk_i)^2} \cdot \mathbf{T}^i + \frac{1}{1+rk_i}\frac{\partial T_j^i}{\partial s_m}\frac{\partial s_m}{\partial x_j} \\
&= -\frac{1}{(1+rk_i)^2}\left(k_i\mathbf{n} + r\frac{\partial k_i}{\partial s_j}\frac{1}{1+rk_j}\mathbf{T}^j \right) \cdot \mathbf{T}^i + \frac{1}{1+rk_i}\frac{\partial \mathbf{T}^i}{\partial s_m} \cdot \frac{1}{1+rk_m}\mathbf{T}^m \\
&= -r\frac{\partial k_i}{\partial s_i}\frac{1}{(1+rk_i)^3}.
\end{aligned}$$

(iii) By (3.4), we have

$$\begin{aligned}
\frac{\partial r}{\partial x_i} &= \left(\delta_{ij} - \frac{\partial \phi_j}{\partial s_m}\frac{\partial s_m}{\partial x_i} \right) N_j + (x_j - \phi_j)\frac{\partial N_j}{\partial s_m}\frac{\partial s_m}{\partial x_i} \\
&= N_i + (x_j - \phi_j)k_m T_j^m \frac{1}{1+rk_m}T_i^m \\
&= N_i + rN_j k_m T_j^m \frac{1}{1+rk_m}T_i^m = N_i.
\end{aligned}$$

That is, $\nabla_x r = \mathbf{n}$. Consequently

$$\Delta_x r = \frac{\partial^2 r}{\partial x_i \partial x_i} = \frac{\partial N_i}{\partial s_j}\frac{\partial s_j}{\partial x_i} = k_j T_i^j \frac{1}{1+rk_j} T_i^j = \sum_{j=1}^{n-1} \frac{k_j}{1+rk_j}.$$

(iv) For any function $v(x,t) = \tilde{v}(s,r,t)$, we have

$$\begin{aligned}
\left(\nabla_x v\right)_i &= \frac{\partial \tilde{v}}{\partial s_j}\frac{\partial s_j}{\partial x_i} + \frac{\partial \tilde{v}}{\partial r}\frac{\partial r}{\partial x_i},\\
\Delta_x v &= \frac{\partial}{\partial x_i}\frac{\partial v}{\partial x_i}\\
&= \frac{\partial s_m}{\partial x_i}\frac{\partial}{\partial s_m}\left(\frac{\partial \tilde{v}}{\partial s_j}\frac{\partial s_j}{\partial x_i} + \frac{\partial \tilde{v}}{\partial r}\frac{\partial r}{\partial x_i}\right) + \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r}\left(\frac{\partial \tilde{v}}{\partial s_j}\frac{\partial s_j}{\partial x_i} + \frac{\partial \tilde{v}}{\partial r}\frac{\partial r}{\partial x_i}\right)\\
&\quad + \frac{\partial \tilde{v}}{\partial s_j}\frac{\partial^2 s_j}{\partial x_i \partial x_i} + \frac{\partial \tilde{v}}{\partial r}\frac{\partial^2 r}{\partial x_i \partial x_i}\\
&= \frac{\partial s_m}{\partial x_i}\left(\frac{\partial^2 \tilde{v}}{\partial s_m \partial s_j}\frac{\partial s_j}{\partial x_i} + \frac{\partial^2 \tilde{v}}{\partial s_m \partial r}\frac{\partial r}{\partial x_i}\right) + \frac{\partial r}{\partial x_i}\left(\frac{\partial^2 \tilde{v}}{\partial r \partial s_j}\frac{\partial s_j}{\partial x_i} + \frac{\partial^2 \tilde{v}}{\partial r^2}\frac{\partial r}{\partial x_i}\right)\\
&\quad + \frac{\partial \tilde{v}}{\partial s_j}\frac{\partial^2 s_j}{\partial x_i \partial x_i} + \frac{\partial \tilde{v}}{\partial r}\frac{\partial^2 r}{\partial x_i \partial x_i}\\
&= \frac{T_i^m}{1+rk_m}\left(\frac{\partial^2 \tilde{v}}{\partial s_m \partial s_j}\frac{1}{1+rk_j}T_i^j + \frac{\partial^2 \tilde{v}}{\partial s_m \partial r}N_i\right)\\
&\quad + N_i\left(\frac{\partial^2 \tilde{v}}{\partial r \partial s_j}\frac{1}{1+rk_j}T_i^j + \frac{\partial^2 \tilde{v}}{\partial r^2}N_i\right)\\
&\quad + \frac{\partial \tilde{v}}{\partial s_j}\left(-r\frac{\partial k_j}{\partial s_j}\frac{1}{(1+rk_j)^3}\right) + \frac{\partial \tilde{v}}{\partial r}\frac{k_j}{1+rk_j}\\
&= \sum_j \frac{1}{(1+rk_j)^2}\frac{\partial^2 \tilde{v}}{\partial s_j^2} + \left(\sum_j \frac{k_j}{1+rk_j}\right)\frac{\partial \tilde{v}}{\partial r} + \frac{\partial^2 \tilde{v}}{\partial r^2}\\
&\quad + \sum_j \frac{\partial \tilde{v}}{\partial s_j}\left(-r\frac{\partial k_j}{\partial s_j}\frac{1}{(1+rk_j)^3}\right). \qquad \square
\end{aligned}$$

The inner expansion requires a rescale of the normal coordinate by $z = r/\varepsilon$. In addition, assume that the parametric representation of the interface has the following expansion:

$$\phi(s,t) = \phi_0(s,t) + \varepsilon\phi_1(s,t) + O(\varepsilon^2).$$

Consequently, the principal curvatures have the following expansions:

$$k_j = k_{j0} + \varepsilon k_{j1} + \varepsilon^2 k_{j2} + \cdots . \tag{3.10}$$

Then by (3.8), Δ_x has the following expansion in ε:

$$\Delta_x = \varepsilon^{-2}\partial_{zz} + \varepsilon^{-1}\kappa_0\partial_z + \Delta_0 + \varepsilon\Delta_1 + O(\varepsilon^2), \tag{3.11}$$

where

$$\Delta_0 := \Delta_s + (z\kappa_1 + \kappa_{01})\partial_z \tag{3.12}$$

and

$$\kappa_0 := \sum_{j=1}^{n-1} k_{j0}, \quad \kappa_{01} := \sum_{j=1}^{n-1} k_{j1}, \quad \kappa_1 := -\sum_{j=1}^{n-1} k_{j0}^2.$$

Note that Δ_1 gives the $O(\varepsilon)$ term in the expansion but its explicit form is not needed for our later calculation. κ_0 and κ_{01} are the $O(1)$ and $O(\varepsilon)$ terms in the expansion of the mean curvature of Γ. When $n = 3$, $\kappa_1 = 2k_{10}k_{20} - (k_{10} + k_{20})^2 = 2K - \kappa_0^2$, where K is the leading order of the Gaussian curvature of Γ.

Now we are ready for the inner expansion. Let $u(x,t) = \tilde{u}(s,z,t)$ and $\mu(x,t) = \tilde{\mu}(s,z,t)$ be expanded as

$$\begin{aligned} u(x,t) &= \tilde{u}_0(s,z,t) + \varepsilon\tilde{u}_1(s,z,t) + \varepsilon^2\tilde{u}_2(s,z,t) + \varepsilon^3\tilde{u}_3(s,z,t) + \cdots, \\ \mu(x,t) &= \tilde{\mu}_0(s,z,t) + \varepsilon\tilde{\mu}_1(s,z,t) + \varepsilon^2\tilde{\mu}_2(s,z,t) + \varepsilon^3\tilde{\mu}_3(s,z,t) + \cdots. \end{aligned}$$

Here $\tilde{u}_0$ is the leading order transition profile that decays exponentially to $u^\pm = \pm 1$ as $z \to \pm\infty$. Then

$$\Delta_x u = \varepsilon^{-2}\tilde{u}_{0zz} + \varepsilon^{-1}\left(\tilde{u}_{1zz} + \kappa_0\tilde{u}_{0z}\right) + \left(\tilde{u}_{2zz} + \kappa_0\tilde{u}_{1z} + \Delta_0\tilde{u}_0\right) + \varepsilon\Delta_1\tilde{u}_0 + O(\varepsilon^2).$$

By (1.2),

$$\begin{aligned} \mu(x,t) = -\varepsilon^2\Delta_x u + F'(u) &= \left(-\tilde{u}_{0zz} + F'(\tilde{u}_0)\right) + \varepsilon\left(-\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + F''(\tilde{u}_0)\tilde{u}_1\right) \\ &+ \varepsilon^2\left(-\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - \Delta_0\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2\right) + O(\varepsilon^3). \end{aligned}$$

So

$$\tilde{\mu}_0 = -\tilde{u}_{0zz} + F'(\tilde{u}_0), \tag{3.13}$$

$$\tilde{\mu}_1 = -\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + F''(\tilde{u}_0)\tilde{u}_1, \tag{3.14}$$

$$\tilde{\mu}_2 = -\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - \Delta_0\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2, \tag{3.15}$$

$$\begin{aligned} \tilde{\mu}_3 = &-\tilde{u}_{3zz} - \kappa_0\tilde{u}_{2z} - \Delta_0\tilde{u}_1 - \Delta_1\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_3 + F'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 \\ &+ \frac{1}{6}F^{(4)}(\tilde{u}_0)\tilde{u}_1^3. \end{aligned} \tag{3.16}$$

Moreover,

$$(\nabla_x\mu)_i = \frac{\partial\tilde{\mu}}{\partial s_j}\frac{\partial s_j}{\partial x_i} + \varepsilon^{-1}\tilde{\mu}_z\frac{\partial\rho}{\partial x_i}$$

and

$$\begin{aligned}\nabla_x \cdot (M(u)\nabla_x \mu) &= M'(\tilde{u})\nabla_x \tilde{u} \cdot \nabla_x \mu + M(\tilde{u})\Delta_x \tilde{\mu} \\ &= \left(\frac{\partial \tilde{u}}{\partial s_k}\frac{\partial s_k}{\partial x_i} + \varepsilon^{-1}\tilde{u}_z \frac{\partial \rho}{\partial x_i}\right)\left(\frac{\partial \tilde{\mu}}{\partial s_j}\frac{\partial s_j}{\partial x_i} + \varepsilon^{-1}\tilde{\mu}_z \frac{\partial \rho}{\partial x_i}\right) \\ &\quad + (1+\tilde{u})(\varepsilon^{-2}\tilde{\mu}_{zz} + \varepsilon^{-1}\kappa_0 \tilde{\mu}_z + \Delta_0 \tilde{\mu} + \varepsilon \Delta_1 \tilde{\mu} + O(\varepsilon^2)) \\ &= \left(\sum_{j=1}^{n-1} \frac{\partial \tilde{u}}{\partial s_j}\frac{\partial \tilde{\mu}}{\partial s_j}\frac{1}{(1+\varepsilon z k_j)^2} + \varepsilon^{-2}\tilde{u}_z \tilde{\mu}_z\right) \\ &\quad + (1+\tilde{u})(\varepsilon^{-2}\tilde{\mu}_{zz} + \varepsilon^{-1}\kappa_0 \tilde{\mu}_z + \Delta_0 \tilde{\mu} + \varepsilon \Delta_1 \tilde{\mu} + O(\varepsilon^2))\,.\end{aligned}$$

Here special attention is needed for $M(\tilde{u}) = 1 + \tilde{u} = 1 + \tilde{u}_0 + \varepsilon \tilde{u}_1 + \varepsilon^2 \tilde{u}_2 + \cdots$. Since $\tilde{u}_0(z)$ decays exponentially to -1 as $z \to -\infty$, $1 + \tilde{u}_0$ is not an $O(1)$ quantity for all $z \in (-\infty, +\infty)$. Indeed assuming $1 + \tilde{u}_0 \sim e^{z/\sigma}$ for $z \to -\infty$, then by taking $\eta = \sigma \ln \frac{1}{\varepsilon}$, we have

$$1 + \tilde{u}_0 \le \begin{cases} O(\varepsilon) & \text{if } z \le -\eta, \\ O(\varepsilon^2) & \text{if } z \le -2\eta, \\ O(\varepsilon^3) & \text{if } z \le -3\eta, \\ O(\varepsilon^4) & \text{if } z \le -4\eta. \end{cases} \tag{3.17}$$

Now it is reasonable to divide $(-\infty, +\infty)$ into subintervals

$$(-\infty, -4\eta), [-4\eta, -3\eta), [-3\eta, -2\eta), [-2\eta, -\eta), [-\eta, +\infty). \tag{3.18}$$

Letting $\chi_4, \chi_3, \chi_2, \chi_1, \chi_0$ be the characteristic functions of these sets, we have the following expansion of $1 + \tilde{u}_0$:

$$\begin{aligned}1 + \tilde{u}_0 &= (1+\tilde{u}_0)\chi_0 + \varepsilon(1+\tilde{u}_0)\chi_1 \varepsilon^{-1} + \varepsilon^2(1+\tilde{u}_0)\chi_2 \varepsilon^{-2} + \varepsilon^3(1+\tilde{u}_0)\chi_3 \varepsilon^{-3} \\ &\quad + \varepsilon^4(1+\tilde{u}_0)\chi_4 \varepsilon^{-4}.\end{aligned} \tag{3.19}$$

The first four terms on the right-hand side are respectively of orders $1, \varepsilon, \varepsilon^2, \varepsilon^3$, and the last one is a residual term of order ε^4 and higher.

Since $\tilde{u}_{0z}$ also decays exponentially to 0 as $z \to \pm\infty$, we expect a similar expansion of $\tilde{u}_{0z}$. Notice that when $z \to -\infty$ it decays in the same rate as $1 + \tilde{u}_0$ decays to 0. As for when $z \to +\infty$, assuming $\tilde{u}_{0z} \sim e^{-z/\hat{\sigma}}$, then $\tilde{u}_{0z} \le O(\varepsilon)$ when $z \ge \hat{\eta} = \hat{\sigma} \ln \frac{1}{\varepsilon}$. So we have the following partition:

$$\begin{aligned}&[-\eta, \hat{\eta}), \quad [-2\eta, -\eta) \cup [\hat{\eta}, 2\hat{\eta}), \quad [-3\eta, -2\eta) \cup [2\hat{\eta}, 3\hat{\eta}), \\ &[-4\eta, -3\eta) \cup [3\hat{\eta}, 4\hat{\eta}), \quad (-\infty, -4\eta) \cup [4\hat{\eta}, +\infty).\end{aligned} \tag{3.20}$$

Letting $\hat{\chi}_0, \hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3, \hat{\chi}_4$ be the characteristic functions, respectively, we have the following expansion of $\tilde{u}_{0z}$:

$$\tilde{u}_{0z} = \hat{\chi}_0 \tilde{u}_{0z} + \varepsilon \hat{\chi}_1 \varepsilon^{-1} \tilde{u}_{0z} + \varepsilon^2 \hat{\chi}_2 \varepsilon^{-2} \tilde{u}_{0z} + \varepsilon^3 \hat{\chi}_3 \varepsilon^{-3} \tilde{u}_{0z} + \varepsilon^4 \hat{\chi}_4 \varepsilon^{-4} \tilde{u}_{0z}. \tag{3.21}$$

Now we can take further expansions of $\nabla_x \cdot (M(u)\nabla_x \mu)$:

$$
\begin{aligned}
&\nabla_x \cdot (M(u)\nabla_x \mu)\\
&= \varepsilon^{-2}\Big\{\chi_0(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_0 \tilde{u}_{0z}\tilde{\mu}_{0z}\Big\}\\
&\quad + \varepsilon^{-1}\Big\{\chi_1\varepsilon^{-1}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_0(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_1\tilde{\mu}_{0zz}\\
&\qquad\quad + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{1z} + \tilde{u}_{1z}\tilde{\mu}_{0z}\Big\}\\
&\quad + \Big\{\chi_2\varepsilon^{-2}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_2\varepsilon^{-2}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_1\varepsilon^{-1}(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z}\\
&\qquad + \tilde{\mu}_{1zz}) + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{1z} + \chi_0(1+\tilde{u}_0)(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \tilde{u}_1(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz})\\
&\qquad + \tilde{u}_2\tilde{\mu}_{0zz} + \frac{\partial \tilde{u}_0}{\partial s_j}\frac{\partial \tilde{\mu}_0}{\partial s_j} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{2z} + \tilde{u}_{1z}\tilde{\mu}_{1z} + \tilde{u}_{2z}\tilde{\mu}_{0z}\Big\}\\
&\quad + \varepsilon\Big\{\chi_3\varepsilon^{-3}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_3\varepsilon^{-3}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_2\varepsilon^{-2}(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz})\\
&\qquad + \hat{\chi}_2\varepsilon^{-2}\tilde{u}_{0z}\tilde{\mu}_{1z} + \chi_1\varepsilon^{-1}(1+\tilde{u}_0)(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{2z}\\
&\qquad + \chi_0(1+\tilde{u}_0)(\Delta_1\tilde{\mu}_0 + \Delta_0\tilde{\mu}_1 + \kappa_0\tilde{\mu}_{2z} + \tilde{\mu}_{3zz}) + \tilde{u}_1(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz})\\
&\qquad + \tilde{u}_2(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_3\tilde{\mu}_{0zz} + \frac{\partial \tilde{u}_0}{\partial s_j}\frac{\partial \tilde{\mu}_1}{\partial s_j} + \frac{\partial \tilde{u}_1}{\partial s_j}\frac{\partial \tilde{\mu}_0}{\partial s_j} - 2z\sum_{j=1}^{n-1} k_{j0}\frac{\partial \tilde{u}_0}{\partial s_j}\frac{\partial \tilde{\mu}_0}{\partial s_j}\\
&\qquad + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{3z} + \tilde{u}_{1z}\tilde{\mu}_{2z} + \tilde{u}_{2z}\tilde{\mu}_{1z} + \tilde{u}_{3z}\tilde{\mu}_{0z}\Big\} + O(\varepsilon^2).
\end{aligned}
\tag{3.22}
$$

4. Time regime $T_2 = t/\varepsilon^2$: Equilibration of the inner structure. In this section we consider the very short time behavior of the Cahn–Hilliard equation (1.1)–(1.2) with the double well potential (1.4) and the highly disparate diffusion mobility $M(u) = 1 + u$.

4.1. Outer expansion. Consider the outer expansion far from the front

$$
\begin{aligned}
u(x,t) &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots,\\
\mu(x,t) &= \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \cdots,
\end{aligned}
$$

where

$$u_i = u_i(x, T_2), \quad \mu_i = \mu_i(x, T_2), \quad T_2 = t/\varepsilon^2.$$

Then by (1.2) and the expansion of $F'(u)$ around u_0, we obtain

$$\mu_0 = F'(u_0), \quad \mu_1 = F''(u_0)u_1, \quad \mu_2 = F''(u_0)u_2 + \frac{1}{2}F'''(u_0)u_1^2 - \Delta u_0.$$

Now since $\partial_t = \varepsilon^{-2}\partial_{T_2}$, we have

$$\partial_t u = \varepsilon^{-2}\partial_{T_2}u_0 + \varepsilon^{-1}\partial_{T_2}u_1 + \partial_{T_2}u_2 + \varepsilon\partial_{T_2}u_3 + \cdots$$

and

$$\nabla\cdot\big(M(u)\nabla\mu\big)=\nabla\cdot\big((1+u_0)\nabla\mu_0\big)+\varepsilon\nabla\cdot\big((1+u_0)\nabla\mu_1+u_1\nabla\mu_0\big)$$
$$+\varepsilon^2\nabla\cdot\Big((1+u_0)\nabla\mu_2+u_1\nabla\mu_1+u_2\nabla\mu_0\Big)$$
$$+\varepsilon^3\nabla\cdot\Big((1+u_0)\nabla\mu_3+u_1\nabla\mu_2+u_2\nabla\mu_1+u_3\nabla\mu_0\Big)+\cdots. \tag{4.1}$$

From (1.1), by matching the terms for every order in ε, we have

$$\partial_{T_2}u_0=0,\quad \partial_{T_2}u_1=0,$$
$$\partial_{T_2}u_2=\nabla\cdot\big((1+u_0)\nabla\mu_0\big),$$
$$\partial_{T_2}u_3=\nabla\cdot\big((1+u_0)\nabla\mu_1+u_1\nabla\mu_0\big).$$

In the T_2 time scale, the initial values are preserved at leading and first orders.

4.2. Inner expansion. Assume u,μ, and ρ have the expansion

$$u(x,t)=\tilde u(s,z,T_2)=\tilde u_0+\varepsilon\tilde u_1+\varepsilon^2\tilde u_2+\varepsilon^3\tilde u_3+\cdots,$$
$$\mu(x,t)=\tilde\mu(s,z,T_2)=\tilde\mu_0+\varepsilon\tilde\mu_1+\varepsilon^2\tilde\mu_2+\varepsilon^3\tilde\mu_3+\cdots,$$
$$\rho(x,T_2)=\rho_0+\varepsilon\rho_1+\varepsilon^2\rho_2+\varepsilon^3\rho_3\ldots;$$

we then have

$$u_t=\varepsilon^{-2}\tilde u_{T_2}+\varepsilon^{-2}\nabla_s\tilde u\cdot\frac{\partial s}{\partial T_2}+\varepsilon^{-3}\frac{\partial\rho}{\partial T_2}\tilde u_z,$$
$$=\varepsilon^{-3}\frac{\partial\rho_0}{\partial T_2}\tilde u_{0z}\hat\chi_0+\varepsilon^{-2}\Bigg(\frac{\partial\rho_0}{\partial T_2}\tilde u_{0z}\hat\chi_1\varepsilon^{-1}+\frac{\partial\rho_1}{\partial T_2}\tilde u_{0z}\hat\chi_0+\frac{\partial\rho_0}{\partial T_2}\tilde u_{1z}+\tilde u_{0T_2}$$
$$+\nabla_s\tilde u_0\cdot\frac{\partial s}{\partial T_2}\Bigg)+O(\varepsilon^{-1}). \tag{4.2}$$

By (1.1), matching (4.2) and (3.22), the ε^{-3} term gives

$$\frac{\partial\rho_0}{\partial T_2}\tilde u_{0z}\hat\chi_0=0,$$

which implies that

$$\frac{\partial\rho_0}{\partial T_2}=0 \tag{4.3}$$

since $\tilde u_0$ is not constant in z. Equation (4.3) indicates that the interface $\Gamma(t)$ does not move and nothing happens in the leading order in the time scale $T_2=t/\varepsilon^2$.

For the ε^{-2} term, we have

$$\frac{\partial\rho_0}{\partial T_2}\tilde u_{0z}\hat\chi_1\varepsilon^{-1}+\frac{\partial\rho_1}{\partial T_2}\tilde u_{0z}\hat\chi_0+\frac{\partial\rho_0}{\partial T_2}\tilde u_{1z}+\tilde u_{0T_2}+\nabla_s\tilde u_0\cdot\frac{\partial s}{\partial T_2}$$
$$=\hat\chi_0\frac{\partial}{\partial z}\left((1+\tilde u_0)\tilde\mu_{0z}\right)+(\chi_0-\hat\chi_0)(1+\tilde u_0)\tilde\mu_{0zz}. \tag{4.4}$$

Since we are mainly interested in the long time behavior of the Cahn–Hilliard equation (1.1)–(1.2) with a highly disparate mobility, we will not try to describe the motion of the interface in the T_2 time scale. Rather we will only consider the shape of the interface when the equilibration is largely completed. The equilibrium state of (4.4) is the steady state of the one-dimensional Cahn–Hilliard equation

$$\frac{\partial}{\partial z}\left((1+\tilde{u}_0)\tilde{\mu}_{0z}\right) = 0 \quad \text{for } z \in (-\eta, \hat{\eta}), \tag{4.5}$$

$$(1+\tilde{u}_0)\tilde{\mu}_{0zz} = 0 \quad \text{for } z \in [\hat{\eta}, +\infty) \tag{4.6}$$

with the boundary conditions

$$\tilde{u}_0 \to u_0^{\pm} = \pm 1 \text{ exponentially as } z \to \pm\infty.$$

The position of the interface is determined by the condition $\tilde{u}_0(0) = 0$.

To find the solution, (4.6) indicates that $\tilde{\mu}_0$ is a linear function of $z \in [\hat{\eta}, +\infty)$. The boundary condition at $z = +\infty$ requires $\tilde{\mu}_0 = 0$.

Equation (4.5) indicates that there is a constant a such that

$$(1+\tilde{u}_0)\tilde{\mu}_{0z} = a \quad \text{for } z \in (-\eta, \hat{\eta}).$$

By the continuity of $\tilde{\mu}_0$ at $z = \hat{\eta}$, we see that $a = 0$. Hence $\tilde{\mu}_0 = b$ in $(-\eta, \hat{\eta})$ for some constant b. Again by continuity we have $b = 0$. So $\tilde{\mu}_0 = 0$ for $z \in (-\eta, +\infty)$, which, by (3.13), translates into an equation for $\tilde{u}_0$:

$$-\tilde{u}_{0zz} + F'(\tilde{u}_0) = 0 \quad \text{for } z \in (-\eta, +\infty), \tag{4.7}$$

$$\tilde{u}_0 \to 1 \text{ as } z \to +\infty, \quad \tilde{u}_0(0) = 0. \tag{4.8}$$

One solution of the above system is the well-known profile $\tilde{u}_0 = U(z) = \tanh(\frac{z}{\sqrt{2}})$. Since $U(z)$ automatically also satisfies our requirements for $\tilde{u}_0$ as $z \to -\infty$, it is natural to assume $\tilde{u}_0 = U(z)$ for all z.

Similar calculation shows that the equilibration process extends to the time scale $T_1 = t/\varepsilon$.

5. The time scale $t = O(1)$: A one-sided Stefan problem.

5.1. Outer expansion. In this time scale,

$$\partial_t u = \partial_t u_0 + \varepsilon \partial_t u_1 + \varepsilon^2 \partial_t u_2 + \varepsilon^3 \partial_t u_3 + \cdots . \tag{5.1}$$

From (1.1), by matching the $O(1)$ terms of (5.1) and (4.1), we obtain a diffusion equation for u_0.

$$\partial_t u_0 = \nabla \cdot \big((1+u_0)\nabla \mu_0\big), \quad \mu_0 = F'(u_0). \tag{5.2}$$

5.2. Inner expansion. For the inner expansion, we have

$$\begin{aligned} u(x,t) &= \tilde{u}(s,z,t) = \tilde{u}_0 + \varepsilon \tilde{u}_1 + \varepsilon^2 \tilde{u}_2 + \varepsilon^3 \tilde{u}_3 + \cdots , \\ \mu(x,t) &= \tilde{\mu}(s,z,t) = \tilde{\mu}_0 + \varepsilon \tilde{\mu}_1 + \varepsilon^2 \tilde{\mu}_2 + \varepsilon^3 \tilde{\mu}_3 + \cdots . \end{aligned} \tag{5.3}$$

Then,

$$u_t = \tilde{u}_t + \nabla_s \tilde{u} \cdot \frac{\partial s}{\partial t} + \varepsilon^{-1} \frac{\partial \rho}{\partial t} \tilde{u}_z = \varepsilon^{-1} \frac{\partial \rho_0}{\partial t} \tilde{u}_{0z} \hat{\chi}_0 + O(1). \tag{5.4}$$

Matching (5.4) and (3.22), the ε^{-2} and ε^{-1} terms give

$$0 = \hat{\chi}_0 \frac{\partial}{\partial z}\left((1+\tilde{u}_0)\tilde{\mu}_{0z}\right) + (\chi_0 - \hat{\chi}_0)(1+\tilde{u}_0)\tilde{\mu}_{0zz}, \tag{5.5}$$

$$\begin{aligned}\frac{\partial \rho_0}{\partial t}\tilde{u}_{0z}\hat{\chi}_0 &= \chi_1 \varepsilon^{-1}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_0(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) \\ &\quad + \tilde{u}_1\tilde{\mu}_{0zz} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{1z} + \tilde{u}_{1z}\tilde{\mu}_{0z}.\end{aligned} \tag{5.6}$$

Equation (5.5) indicates that the leading order of the transition layer is equilibrated. Hence $\tilde{u}_0 = U(z)$ and $\tilde{\mu}_0 = -U''(z) + F'(U) = 0$. So (5.6) can be simplified into

$$\begin{aligned}\frac{\partial \rho_0}{\partial t}U'(z)\hat{\chi}_0 &= \chi_0(1+U)\tilde{\mu}_{1zz} + \hat{\chi}_0 U'\tilde{\mu}_{1z} \\ &= \chi_0 \frac{\partial}{\partial z}\Big((1+U)\tilde{\mu}_{1z}\Big) - (\chi_0 - \hat{\chi}_0)U'\tilde{\mu}_{1z}.\end{aligned} \tag{5.7}$$

Recall that $\tilde{\mu}_{1z} = O(1)$, $U'(z) \le O(\varepsilon)$ for $z \ge \hat{\eta}$, and that $U'(z)$ decays exponentially to 0 as $z \to +\infty$. Integrating (5.7) in z from $-\infty$ to $+\infty$, we have

$$\frac{\partial \rho_0}{\partial t}U\Big|_{-\eta}^{\hat{\eta}} = (1+U)\tilde{\mu}_{1z}\Big|_{-\eta}^{+\infty} + O(\varepsilon). \tag{5.8}$$

Since $1 - O(\varepsilon) \le U(\hat{\eta}) \le 1, -1 \le U(-\eta) \le -1 + O(\varepsilon)$, (5.8) can be interpreted as

$$\frac{\partial \rho_0}{\partial t}([U] + O(\varepsilon)) = 2\lim_{z\to+\infty}\tilde{\mu}_{1z} + O(\varepsilon). \tag{5.9}$$

Here $[U] = u^+ - u^- = 2$.

To get $\lim_{z\to+\infty}\tilde{\mu}_{1z}$, we need a match condition between the inner and the outer expansions. As in [13], for fixed $x \in \Gamma$, we require

$$(\mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots)(x + \varepsilon z\mathbf{n}, t) \approx (\tilde{\mu}_0 + \varepsilon\tilde{\mu}_1 + \varepsilon^2\tilde{\mu}_2 + \cdots)(s, z, t) \tag{5.10}$$

when εz is between $O(\varepsilon)$ and $o(1)$. Expanding the left-hand side around x as $\varepsilon z \to 0^+$, we have

$$\mu_0^+ + \varepsilon(\mu_1^+ + z\partial_{\mathbf{n}}\mu_0^+) + \varepsilon^2\left(\mu_2^+ + z\partial_{\mathbf{n}}\mu_1^+ + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_0^+\right) + \cdots,$$

where $\partial_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$, and

$$\mu_i^+(x,t) = \lim_{s\to 0^+}\mu_i(x + s\mathbf{n}, t) \quad \text{for all } i.$$

We can obtain a similar expansion as $\varepsilon z \to 0^-$. The match condition (5.10) gives

$$\mu_0^\pm = \lim_{z\to\pm\infty}\tilde{\mu}_0, \tag{5.11}$$

$$\mu_1^\pm + z\partial_{\mathbf{n}}\mu_0^\pm = \tilde{\mu}_1 + o(1) \quad \text{as } z \to \pm\infty, \tag{5.12}$$

$$\mu_2^\pm + z\partial_{\mathbf{n}}\mu_1^\pm + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_0^\pm = \tilde{\mu}_2 + o(1) \quad \text{as } z \to \pm\infty. \tag{5.13}$$

Similarly we obtain match conditions for u,

$$u_0^{\pm} = \lim_{z\to\pm\infty} \tilde{u}_0, \tag{5.14}$$

$$u_1^{\pm} + z\partial_{\mathbf{n}} u_0^{\pm} = \tilde{u}_1 + o(1) \quad \text{as } z \to \pm\infty, \tag{5.15}$$

$$u_2^{\pm} + z\partial_{\mathbf{n}} u_1^{\pm} + \frac{1}{2} z^2 \partial_{\mathbf{n}}^2 u_0^{\pm} = \tilde{u}_2 + o(1) \quad \text{as } z \to \pm\infty. \tag{5.16}$$

By match condition (5.12), $\tilde{\mu}_{1z} \to \partial_{\mathbf{n}}\mu_0^+$ as $z \to +\infty$. So in the limit equation (5.9) becomes

$$V = -\frac{\partial \rho_0}{\partial t} = -\partial_{\mathbf{n}}\mu_0^+. \tag{5.17}$$

5.3. Sharp interface limit in the time scale *t*. Combining (5.2), (5.17) and the match condition (5.14), formally in the sharp interface limit, we have

$$\partial_t u_0 = \nabla\cdot\big((1+u_0)\nabla\mu_0\big), \quad \mu_0 = F'(u_0) \quad \text{in } \Omega_\pm, \tag{5.18}$$

$$u_0^{\pm} = \pm 1 \quad \text{on } \Gamma, \tag{5.19}$$

$$V = -\partial_{\mathbf{n}}\mu_0^+. \tag{5.20}$$

The velocity of the interface is determined by data in Ω_+. This is a one-sided Stefan problem.

6. The time scale $t_1 = \varepsilon t$: A one-sided Hele–Shaw problem.

6.1. Outer expansion. In the time scale $t_1 = \varepsilon t$,

$$\partial_t u = \varepsilon\partial_{t_1} u_0 + \varepsilon^2\partial_{t_1} u_1 + \varepsilon^3\partial_{t_1} u_2 + \cdots. \tag{6.1}$$

Matching the terms of (6.1) and (4.1) for every order in ε, we have

$$0 = \nabla\cdot\big((1+u_0)\nabla\mu_0\big), \quad \mu_0 = F'(u_0), \tag{6.2}$$

$$\partial_{t_1} u_0 = \nabla\cdot\big((1+u_0)\nabla\mu_1 + u_1\nabla\mu_0\big), \tag{6.3}$$

$$\partial_{t_1} u_1 = \nabla\cdot\Big((1+u_0)\nabla\mu_2 + u_1\nabla\mu_1 + u_2\nabla\mu_0\Big). \tag{6.4}$$

Equation (6.2) indicates that u_0 and consequently μ_0 are in steady states. So it is reasonable to assume the solution for (6.2) is

$$u_0 = \begin{cases} 1 & \text{in } \Omega_+, \\ -1 & \text{in } \Omega_-. \end{cases}$$

Then $\mu_0 = F'(u_0) = 0$. Since $\mu_1 = F''(u_0)u_1 = 2u_1$, (6.3) translates into

$$-\Delta\mu_1 = 0 \text{ in } \Omega_+. \tag{6.5}$$

In Ω_-, since $u_0 = -1$, $\mu_0 = 0$, we cannot get any useful information from (6.3). Instead μ_1 must be determined by (6.4). That is, u_1 satisfies a nonlinear diffusion equation in Ω_-,

$$\partial_{t_1} u_1 = \nabla\cdot(u_1\nabla\mu_1) \text{ in } \Omega_-. \tag{6.6}$$

Since $\mu_1 = F''(u_0)u_1 = 2u_1$, it turns out μ_1 satisfies a porous medium equation

$$\partial_{t_1}\mu_1 = \nabla\cdot(\mu_1\nabla\mu_1) \text{ in } \Omega_-. \tag{6.7}$$

6.2. Inner expansion. For the inner expansion, we have

$$\begin{aligned} u(x,t) &= \tilde{u}(s,z,t) = \tilde{u}_0 + \varepsilon\tilde{u}_1 + \varepsilon^2\tilde{u}_2 + \varepsilon^3\tilde{u}_3 + \cdots, \\ \mu(x,t) &= \tilde{\mu}(s,z,t) = \tilde{\mu}_0 + \varepsilon\tilde{\mu}_1 + \varepsilon^2\tilde{\mu}_2 + \varepsilon^3\tilde{\mu}_3 + \cdots, \\ \rho(x,t) &= \rho_0 + \varepsilon\rho_1 + \cdots; \end{aligned}$$

then,

$$\begin{aligned} u_t = \varepsilon\left(\tilde{u}_{t_1} + \nabla_s\tilde{u}\cdot\frac{\partial s}{\partial t_1}\right) + \frac{\partial\rho}{\partial t_1}\tilde{u}_z = \frac{\partial\rho_0}{\partial t_1}\tilde{u}_{0z}\hat{\chi}_0 \\ + \varepsilon\left(\frac{\partial\rho_1}{\partial t_1}\tilde{u}_{0z}\hat{\chi}_0 + \frac{\partial\rho_0}{\partial t_1}\tilde{u}_{1z} + \frac{\partial\rho_0}{\partial t_1}\tilde{u}_{0z}\hat{\chi}_1\varepsilon^{-1} + \tilde{u}_{0t_1} + \nabla_s\tilde{u}_0\cdot\frac{\partial s}{\partial t_1}\right) + O(\varepsilon^2). \end{aligned} \tag{6.8}$$

Matching (6.8) and (3.22), the ε^{-2}, ε^{-1}, and ε^0 terms give

$$0 = \chi_0(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{0z}, \tag{6.9}$$

$$\begin{aligned} 0 = \chi_1\varepsilon^{-1}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_0(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_1\tilde{\mu}_{0zz} \\ + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{1z} + \tilde{u}_{1z}\tilde{\mu}_{0z}, \end{aligned} \tag{6.10}$$

$$\begin{aligned} \frac{\partial\rho_0}{\partial t_1}\tilde{u}_{0z}\hat{\chi}_0 = {} & \chi_2\varepsilon^{-2}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_2\varepsilon^{-2}\tilde{u}_{0z}\tilde{\mu}_{0z} \\ & + \chi_1\varepsilon^{-1}(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{1z} \\ & + \chi_0(1+\tilde{u}_0)(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \tilde{u}_1(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_2\tilde{\mu}_{0zz} \\ & + \frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{2z} + \tilde{u}_{1z}\tilde{\mu}_{1z} + \tilde{u}_{2z}\tilde{\mu}_{0z}. \end{aligned} \tag{6.11}$$

Equation (6.9) indicates that the interface is stable in the leading order, that is, $\tilde{u}_0(s,z,t) = U(z)$ and $\tilde{\mu}_0 = 0$.

Equation (6.10) is simplified into

$$\hat{\chi}_0\frac{\partial}{\partial z}\left((1+\tilde{u}_0)\tilde{\mu}_{1z}\right) + (\chi_0 - \hat{\chi}_0)(1+\tilde{u}_0)\tilde{\mu}_{1zz} = 0.$$

This means

$$\frac{\partial}{\partial z}\left((1+\tilde{u}_0)\tilde{\mu}_{1z}\right) = 0 \quad \text{if } z\in(-\eta,\hat{\eta}), \tag{6.12}$$

$$(1+\tilde{u}_0)\tilde{\mu}_{1zz} = 0 \quad \text{if } z\in[\hat{\eta},+\infty). \tag{6.13}$$

Equation (6.13) indicates that $\tilde{\mu}_1$ is a linear function of z in $[\hat{\eta},+\infty)$. By match condition (5.12), we have $\lim_{z\to+\infty}\tilde{\mu}_{1z} = \partial_{\mathbf{n}}\mu_0^+ = 0$. So there exists $c(s,t)$, independent of z, such that $\tilde{\mu}_1 = c$ for $z\in[\hat{\eta},+\infty)$.

As for $z\in(-\eta,\hat{\eta})$, (6.12) indicates that there exists $b(s,t)$ independent of z such that $(1+U)\tilde{\mu}_{1z} = b$. The smooth continuity at $z=\hat{\eta}$ requires $b=0$. Consequently

$\tilde{\mu}_1$ is a constant in $(-\eta, \hat{\eta})$. Again by continuity at $z = \hat{\eta}$ we see that $\tilde{\mu}_1 = c$ for $z \in (-\eta, +\infty)$.S

About $\tilde{\mu}_1$ for $z \in (-\infty, -\eta)$, since the match condition (5.12) gives $\tilde{\mu}_{1z} \to \partial_{\mathbf{n}} \mu_0^- = 0$ as $z \to -\infty$, it is reasonable to make an ad hoc assumption through extrapolation that $\tilde{\mu}_1 = c$ for all $z \in (-\infty, \infty)$. So

$$c = \tilde{\mu}_1 = -\tilde{u}_{1zz} - \kappa_0 \tilde{u}_{0z} + F''(\tilde{u}_0)\tilde{u}_1. \tag{6.14}$$

Define an operator $\mathcal{L} := -\partial_{zz} + F''(U)$. Then (6.14) can be rewritten as

$$\kappa_0 \tilde{u}_{0z} + c = \mathcal{L}\tilde{u}_1. \tag{6.15}$$

To find c, we apply a solvability condition of (6.15). Namely, $\kappa_0 \tilde{u}_{0z} + c$ need to be perpendicular to the kernel of the self-adjoint operator $\mathcal{L}$ spanned by $\tilde{u}_{0z} = U'$. Multiplying both sides of (6.14) by $\tilde{u}_{0z}$ and integrating in z from $-\infty$ to $+\infty$, we have

$$\begin{aligned}\kappa_0 \int_{-\infty}^{\infty} \tilde{u}_{0z}^2 \, dz + c \int_{-\infty}^{\infty} \tilde{u}_{0z} \, dz &= \int_{-\infty}^{\infty} \Big(-\tilde{u}_{1zz} + F''(\tilde{u}_0)\tilde{u}_1 \Big) \tilde{u}_{0z} \, dz \\ &= \int_{-\infty}^{\infty} \Big(-\tilde{u}_{0zzz} + F''(\tilde{u}_0)\tilde{u}_{0z} \Big) \tilde{u}_1 \, dz \\ &= \int_{-\infty}^{\infty} \tilde{\mu}_{0z} \tilde{u}_1 \, dz = 0.\end{aligned}$$

So

$$\tilde{\mu}_1 = c = -\kappa_0 \frac{S}{[U]}, \tag{6.16}$$

where

$$S = \int_{-\infty}^{\infty} \tilde{u}_{0z}^2 \, dz = \int_{-\infty}^{\infty} U'(z)^2 \, dz.$$

Again by match condition (5.12), we conclude that in the sharp interface limit, the boundary condition for μ_1 on Γ is

$$\mu_1 = -\kappa_0 \frac{S}{[U]} \quad \text{on } \Gamma. \tag{6.17}$$

This is the Gibbs–Thomson condition.

From (6.15) $\tilde{u}_1$ can be solved as $\tilde{u}_1 = \kappa_0 \Phi_1 + \alpha_1 U'(z)$, where $\Phi_1 \perp \operatorname{Ker}\mathcal{L}$ satisfies $\mathcal{L}\Phi_1 = U' - S[U]^{-1}$ and $\alpha_1 U'(z) \in \operatorname{Ker}\mathcal{L}$. Since the position of Γ is chosen so as to $\tilde{u}_1(0) = 0$, we have $\alpha_1 = -\kappa_0 \Phi_1(0)/U'(0)$. That is,

$$\tilde{u}_1 = \kappa_0 \tilde{\Phi}_1, \tag{6.18}$$

where $\tilde{\Phi}_1 := \Phi_1 - U'(z)\Phi_1(0)/U'(0)$ and

$$\tilde{\Phi}_1 \to -\frac{S}{[U]F''(\pm 1)} \quad \text{as } z \to \pm\infty.$$

Equation (6.11) can be simplified into

$$\begin{aligned}\frac{\partial \rho_0}{\partial t_1} U' \hat{\chi}_0 &= \chi_0 (1+U) \tilde{\mu}_{2zz} + \hat{\chi}_0 U' \tilde{\mu}_{2z} \\ &= \chi_0 \frac{\partial}{\partial z}\Big((1+U)\tilde{\mu}_{2z} \Big) - (\chi_0 - \hat{\chi}_0) U' \tilde{\mu}_{2z}.\end{aligned} \tag{6.19}$$

As is done in section 5, integrating in z from $-\infty$ to ∞, we have

$$\frac{\partial \rho_0}{\partial t_1}(2+O(\varepsilon)) = 2\lim_{z\to+\infty}\tilde{\mu}_{2z} + O(\varepsilon) = 2\partial_{\mathbf{n}}\mu_1^+ + O(\varepsilon). \tag{6.20}$$

So

$$V = -\frac{\partial \rho_0}{\partial t_1} = -\partial_{\mathbf{n}}\mu_1^+. \tag{6.21}$$

6.3. Sharp interface limits. In summary, the evolution of the interface Γ is determined, in the leading order, by the following model:

$$\Delta\mu_1 = 0 \quad \text{in } \Omega_+, \tag{6.22}$$

$$\mu_1 = -\frac{S}{2}\kappa_0 \quad \text{on } \Gamma, \tag{6.23}$$

$$V = -\partial_{\mathbf{n}}\mu_1^+ \quad \text{on } \Gamma. \tag{6.24}$$

This sharp interface problem is a one-sided Hele–Shaw problem.

As for the data in Ω_-, we know $u_0 = -1$ and $\mu_1 = F''(u_0)u_1 = 2u_1$. By (6.7) and (6.17), μ_1 needs to satisfy

$$\begin{aligned} \partial_{t_1}\mu_1 &= \nabla\cdot(\mu_1\nabla\mu_1) \quad \text{in } \Omega_-, \\ \mu_1 &= -\frac{S}{2}\kappa_0 \quad \text{on } \Gamma. \end{aligned}$$

The flux from the data in Ω_- contributes to an $O(\varepsilon)$ correction to the normal velocity of the interface, together with the surface diffusion, that is, the surface Laplacian of the mean curvature. We will not work out the details of this $O(\varepsilon)$ correction because it is too complicated and an explicit formula is unlikely to be obtained. We will, however, in section 7 study the interfacial dynamics in the $t_2 = \varepsilon^2 t$ time scale, after the t_1 dynamics is equilibrated. The effect of the flux from Ω_- will be clear.

We will concentrate on the scenario that Ω_+ is a minor phase and hence it breaks into a collection of disjoint small components of various sizes.

7. The time scale $t_2 = \varepsilon^2 t$: Quasi-stationary porous medium equations. In this section, we will keep in mind the scenario that Ω_+ is a minor phase and hence it breaks into a collection of disjoint small components of various sizes. Here ω_+ is such a small component and we denote its boundary by $\gamma = \partial\omega_+$.

Suppose the interfacial dynamics of γ is equilibrated in leading order in the t_1 time scale. By (6.24), it requires $\partial_{\mathbf{n}}\mu_1^+ = 0$ on γ. Combined with (6.22) it indicates that μ_1 is a constant in ω_+. Then by (6.23), γ should be of constant mean curvature in leading order. In two space dimensions this requires ω_+ to be roughly a circle. In three space dimensions the morphology can be more complicated but the simplest one is that ω_+ is roughly spherical.

7.1. Outer expansion. In this case the outer expansion needs to be carried out outside of the transition region around the interface γ. Write

$$\begin{aligned} u(x,t) &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots, \\ \mu(x,t) &= \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \cdots, \end{aligned}$$

where

$$u_i = u_i(x, t_2), \quad \mu_i = \mu_i(x, t_2). \tag{7.1}$$

$$\partial_t u = \varepsilon^2 \partial_{t_2} u_0 + \varepsilon^3 \partial_{t_2} u_1 + \varepsilon^4 \partial_{t_2} u_2 + \cdots . \tag{7.2}$$

By matching the terms of (7.2) and (4.1) for every order in ε, we have

$$0 = \nabla \cdot \big((1 + u_0)\nabla \mu_0\big), \quad \mu_0 = F'(u_0), \tag{7.3}$$

$$0 = \nabla \cdot \big((1 + u_0)\nabla \mu_1 + u_1 \nabla \mu_0\big), \tag{7.4}$$

$$\partial_{t_2} u_0 = \nabla \cdot \Big((1 + u_0)\nabla \mu_2 + u_1 \nabla \mu_1 + u_2 \nabla \mu_0\Big), \tag{7.5}$$

$$\partial_{t_2} u_1 = \nabla \cdot \Big((1 + u_0)\nabla \mu_3 + u_1 \nabla \mu_2 + u_2 \nabla \mu_1 + u_3 \nabla \mu_0\Big). \tag{7.6}$$

As in the previous section, it is reasonable to assume $u_0 = \pm 1$ in $\Omega_\pm$, respectively, and consequently $\mu_0 = F'(u_0) = 0$. Equation (7.4) translates into

$$\nabla \cdot \Big((1 + u_0)\nabla \mu_1\Big) = 0.$$

That is,

$$\Delta \mu_1 = 0 \quad \text{in } \omega_+. \tag{7.7}$$

In Ω_-, u_1 and μ_1 must be determined by (7.5). Since $\mu_1 = F''(u_0)u_1 = (3u_0^2 - 1)u_1 = 2u_1$, we have a quasi-stationary porous medium equation

$$\nabla \cdot (\mu_1 \nabla \mu_1) = 0 \quad \text{in } \Omega_-. \tag{7.8}$$

Now we consider μ_2. In ω_+, by (7.5), since $u_0 = 1$ and $\mu_1 = 2u_1$, we have

$$\Delta \mu_2 = -\frac{1}{4}\nabla \cdot (\mu_1 \nabla \mu_1) = -\frac{1}{4}|\nabla \mu_1|^2 \quad \text{in } \omega_+. \tag{7.9}$$

In Ω_-, since $u_0 = -1, \mu_0 = 0$ and because of (7.8), we cannot get any useful information from (7.5). Instead we need to look into (7.6). Upon simplification, we have

$$\partial_{t_2} u_1 = \nabla \cdot \Big(u_1 \nabla \mu_2 + u_2 \nabla \mu_1\Big) \quad \text{in } \Omega_-.$$

7.2. Inner expansion. In the transition region around γ, we have the following expansion:

$$u(x, t) = \tilde{u}(s, z, t_2) = \tilde{u}_0 + \varepsilon \tilde{u}_1 + \varepsilon^2 \tilde{u}_2 + \varepsilon^3 \tilde{u}_3 + \cdots ,$$
$$\mu(x, t) = \tilde{\mu}(s, z, t_2) = \tilde{\mu}_0 + \varepsilon \tilde{\mu}_1 + \varepsilon^2 \tilde{\mu}_2 + \varepsilon^3 \tilde{\mu}_3 + \cdots ,$$
$$\rho(x, t) = \rho_0 + \varepsilon \rho_1 + \cdots .$$

Then

$$u_t = \varepsilon^2 \left(\tilde{u}_{t_2} + \nabla_s \tilde{u} \cdot \frac{\partial s}{\partial t_2}\right) + \varepsilon \frac{\partial \rho}{\partial t_2}\tilde{u}_z = \varepsilon \frac{\partial \rho_0}{\partial t_2}\tilde{u}_{0z}\hat{\chi}_0 + O(\varepsilon^2). \tag{7.10}$$

By (1.1), matching (7.10) and (3.22), the ε^{-2}, ε^{-1}, ε^0, and ε terms give

$$0 = \chi_0(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{0z}, \tag{7.11}$$

$$\begin{aligned} 0 &= \chi_1\varepsilon^{-1}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_0(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_1\tilde{\mu}_{0zz} \\ &\quad + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{1z} + \tilde{u}_{1z}\tilde{\mu}_{0z}, \end{aligned} \tag{7.12}$$

$$\begin{aligned} 0 &= \chi_2\varepsilon^{-2}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_2\varepsilon^{-2}\tilde{u}_{0z}\tilde{\mu}_{0z} \\ &\quad + \chi_1\varepsilon^{-1}(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{1z} \\ &\quad + \chi_0(1+\tilde{u}_0)(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \tilde{u}_1(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_2\tilde{\mu}_{0zz} \\ &\quad + \frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{2z} + \tilde{u}_{1z}\tilde{\mu}_{1z} + \tilde{u}_{2z}\tilde{\mu}_{0z}, \end{aligned} \tag{7.13}$$

$$\begin{aligned} \frac{\partial\rho_0}{\partial t_2}\tilde{u}_{0z}\hat{\chi}_0 &= \chi_3\varepsilon^{-3}(1+\tilde{u}_0)\tilde{\mu}_{0zz} + \hat{\chi}_3\varepsilon^{-3}\tilde{u}_{0z}\tilde{\mu}_{0z} + \chi_2\varepsilon^{-2}(1+\tilde{u}_0)(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) \\ &\quad + \hat{\chi}_2\varepsilon^{-2}\tilde{u}_{0z}\tilde{\mu}_{1z} + \chi_1\varepsilon^{-1}(1+\tilde{u}_0)(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \hat{\chi}_1\varepsilon^{-1}\tilde{u}_{0z}\tilde{\mu}_{2z} \\ &\quad + \chi_0(1+\tilde{u}_0)(\Delta_1\tilde{\mu}_0 + \Delta_0\tilde{\mu}_1 + \kappa_0\tilde{\mu}_{2z} + \tilde{\mu}_{3zz}) \\ &\quad + \tilde{u}_1(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}) + \tilde{u}_2(\kappa_0\tilde{\mu}_{0z} + \tilde{\mu}_{1zz}) + \tilde{u}_3\tilde{\mu}_{0zz} \\ &\quad + \frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_1}{\partial s_j} + \frac{\partial\tilde{u}_1}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} - 2z\sum_{j=1}^{n-1} k_{j0}\frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} \\ &\quad + \hat{\chi}_0\tilde{u}_{0z}\tilde{\mu}_{3z} + \tilde{u}_{1z}\tilde{\mu}_{2z} + \tilde{u}_{2z}\tilde{\mu}_{1z} + \tilde{u}_{3z}\tilde{\mu}_{0z}. \end{aligned} \tag{7.14}$$

As is done in section 6, (7.11) and (7.12) indicate that

$$\tilde{u}_0 = U(z), \quad \tilde{\mu}_0 = 0, \quad \tilde{u}_1 = \kappa_0\tilde{\Phi}_1, \quad \tilde{\mu}_1 = -\kappa_0\frac{S}{[U]}. \tag{7.15}$$

So by match condition (5.12), we have

$$\mu_1 = -\kappa_0\frac{S}{[U]} \quad \text{on } \gamma. \tag{7.16}$$

Since the t_1 dynamics (6.22)–(6.24) is equilibrated, we have $\partial_{\mathbf{n}}\mu_1^+ = 0$ and μ_1 is a constant in ω_+. Consequently γ is of constant mean curvature in the leading order.

Now we consider $\tilde{\mu}_2$.

LEMMA 7.1. *There exists a constant S_1 such that $\tilde{\mu}_2$ is asymptotically linear with slope $\partial_{\mathbf{n}}\mu_1^-$ as $z \to -\infty$ and*

$$\tilde{\mu}_2 = -\kappa_0^2\frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]} \quad \textit{for } z \in (-\eta, +\infty). \tag{7.17}$$

As for values of $\tilde{\mu}_2$ in $(-\infty, -\eta)$, the exact formula cannot be obtained but we reasonably extrapolate a little bit and assume $\tilde{\mu}_2 = -\kappa_0^2\frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]}$ for $z \in (-2\eta, -\eta)$.

Proof. By (7.13), we have

$$(1+U)\tilde{\mu}_{2zz} = 0 \quad \text{if } z \in [\hat{\eta}, +\infty), \tag{7.18}$$

$$0 = \frac{\partial}{\partial z}\Big((1+U)\tilde{\mu}_{2z}\Big) \quad \text{if } z \in (-\eta, \hat{\eta}). \tag{7.19}$$

Equation (7.18) requires that $\tilde{\mu}_{2z}$ is a constant for $z \in [\hat{\eta}, +\infty)$. Since $\tilde{\mu}_{2z} \to \partial_{\mathbf{n}}\mu_1^+ = 0$ as $z \to +\infty$, we have $\tilde{\mu}_{2z} = 0$ for $z \in [\hat{\eta}, +\infty)$. By (7.19), there is b_2 independent of z

such that

$$b_2 = (1+U)\tilde{\mu}_{2z} \quad \text{for all } z \in (-\eta, \hat{\eta}).$$

By the continuity of $\tilde{\mu}_{2z}$ at $z = \hat{\eta}$, we have $b_2 = 0$. Hence there exists $c_2(s,t)$ independent of z such that

$$\tilde{\mu}_2 = c_2 \quad \text{for } z \in (-\eta, +\infty). \tag{7.20}$$

As for $z \in (-\infty, -\eta)$, we resort to the match condition (5.13), which requires

$$\lim_{z\to-\infty} \tilde{\mu}_{2z} = \partial_{\mathbf{n}}\mu_1^-.$$

Since in Ω_-, μ_1 solves (7.8) and (7.16), in general $\partial_{\mathbf{n}}\mu_1^-$ is not 0. This means $\tilde{\mu}_2$ has a tail as $z \to -\infty$ that is asymptotically linear with a nonzero slope $\partial_{\mathbf{n}}\mu_1^-$.

To find c_2, by (3.15), we have

$$\begin{aligned}
\tilde{\mu}_2 &= -\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - \Delta_0\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2 \\
&= -\tilde{u}_{2zz} + F''(\tilde{u}_0)\tilde{u}_2 - \kappa_0\tilde{u}_{1z} - (\kappa_1 z + \kappa_{01})\tilde{u}_{0z}\hat{\chi}_0 + \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2.
\end{aligned}$$

So

$$\mathcal{L}\tilde{u}_2 = \tilde{\mu}_2 + \kappa_0\tilde{u}_{1z} + (\kappa_1 z + \kappa_{01})\tilde{u}_{0z}\hat{\chi}_0 - \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2. \tag{7.21}$$

The solvability condition requires

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} \left(\tilde{\mu}_2 + \kappa_0\tilde{u}_{1z} + (\kappa_1 z + \kappa_{01})\tilde{u}_{0z}\hat{\chi}_0 - \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2\right) U'\, dz \\
&= \int_{-\infty}^{\infty} \left(c_2 + \kappa_0\tilde{u}_{1z} + (\kappa_1 z + \kappa_{01})\tilde{u}_{0z} - \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2\right) U'\, dz + O(\varepsilon) \\
&= [U]c_2 + \int_{-\infty}^{\infty} \left(\kappa_0\tilde{u}_{1z} - \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2\right) U'\, dz + \kappa_1 \int_{-\infty}^{\infty} zU'^2\, dz \\
&\quad + \kappa_{01} \int_{-\infty}^{\infty} U'^2\, dz + O(\varepsilon) \\
&= [U]c_2 + \int_{-\infty}^{\infty} \left(\kappa_0\tilde{u}_{1z} - \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2\right) U'\, dz + \kappa_{01} \int_{-\infty}^{\infty} U'^2\, dz + O(\varepsilon) \\
&= [U]c_2 + \kappa_0^2 \int_{-\infty}^{\infty} \left(\tilde{\Phi}_1' - \frac{1}{2}F'''(U)\tilde{\Phi}_1^2\right) U'\, dz + \kappa_{01} \int_{-\infty}^{\infty} U'^2\, dz + O(\varepsilon).
\end{aligned}$$

So

$$c_2 = -\kappa_0^2 \frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]}, \tag{7.22}$$

where

$$S_1 = \int_{-\infty}^{\infty} \left(\tilde{\Phi}_1' - \frac{1}{2}F'''(U)\tilde{\Phi}_1^2\right) U'\, dz. \qquad \square \tag{7.23}$$

To calculate the normal velocity, we simplify (7.14) and get

$$\frac{\partial\rho_0}{\partial t_2}\tilde{u}_{0z}\hat{\chi}_0 = \chi_0 \frac{\partial}{\partial z}\left((1+\tilde{u}_0)\tilde{\mu}_{3z}\right) + \frac{\partial}{\partial z}(\tilde{u}_1\tilde{\mu}_{2z}) + (\hat{\chi}_0 - \chi_0)\tilde{u}_{0z}\tilde{\mu}_{3z}. \tag{7.24}$$

Integrating from $-\infty$ to $+\infty$, we have

$$\begin{aligned}\frac{\partial\rho_0}{\partial t_2}([U]+O(\varepsilon)) &= \lim_{z\to+\infty}(2\tilde{\mu}_{3z}+\tilde{u}_1\tilde{\mu}_{2z}) - \lim_{z\to-\infty}\tilde{u}_1\tilde{\mu}_{2z}+O(\varepsilon)\\ &= 2\partial_{\mathbf{n}}\mu_2^+ - u_1^-\partial_{\mathbf{n}}\mu_1^- + O(\varepsilon).\end{aligned} \tag{7.25}$$

So

$$V = -\frac{\partial\rho_0}{\partial t_2} = -\partial_{\mathbf{n}}\mu_2^+ + \frac{1}{4}u_1^-\partial_{\mathbf{n}}\mu_1^-. \tag{7.26}$$

Here we use $u_1^- = \frac{1}{2}\mu_1^-$.

7.3. Sharp interface limit. Since μ_1 is constant in ω_+, (7.9) becomes

$$\Delta\mu_2 = 0 \quad \text{in } \omega_+. \tag{7.27}$$

By match condition (5.13), (7.17) gives

$$\mu_2^+ = -\kappa_0^2\frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]} \quad \text{on } \gamma. \tag{7.28}$$

Combined with (7.8), (7.16), and (7.26), the evolution of the interface $\gamma = \partial\omega_+$ is determined by the following system:

$$\Delta\mu_2 = 0 \quad \text{in } \omega_+, \tag{7.29}$$

$$\mu_2^+ = -\kappa_0^2\frac{S_1}{[U]} - \kappa_{01}\frac{S}{[U]} \quad \text{on } \gamma, \tag{7.30}$$

$$\nabla\cdot(\mu_1\nabla\mu_1) = 0 \quad \text{in } \Omega_-, \tag{7.31}$$

$$\mu_1 = -\frac{S}{[U]}\kappa_0 \quad \text{on } \gamma, \tag{7.32}$$

$$V = -\partial_{\mathbf{n}}\mu_2^+ + \frac{1}{4}\mu_1^-\partial_{\mathbf{n}}\mu_1^- \quad \text{on } \gamma. \tag{7.33}$$

8. Discussion. We formally derive the sharp interface models for different time scales for the Cahn–Hilliard equation with highly disparate diffusion mobility. In the time scale $O(\varepsilon^{-1})$ the diffusion process in the phase with constant nonzero mobility dominates, resulting in a one-sided model. In the slower time scale $O(\varepsilon^{-2})$ a porous medium diffusion process in the phase with degenerate mobility is the dominating mechanism for coarsening. These sharp interface models behave dramatically differently from the Mullins–Sekerka model, which corresponds to the Cahn–Hilliard equation with a mobility that is constant, or the surface diffusion model, which is believed to correspond to the Cahn–Hilliard equation with a mobility that is degenerate in both phases.

The existence of a porous medium diffusion process in the phase with degenerate mobility is nontrivial. Indeed this is the mechanism for the coarsening to occur. The analysis for our model indicates that for the case when the diffusion mobility is degenerate in both phases, porous medium diffusion processes should also exist, as suggested by heuristic arguments in [2]. It should be the combination of surface diffusion and the porous medium diffusion processes in both phases that gives rise to the coarsening phenomenon. The investigation is in progress and will be reported separately.

While our analysis is focused on the model Cahn–Hilliard equation (1.2), the findings and approaches may also be applicable to realistic thin liquid films ([1, 12] and references therein) and multiphase systems with highly disparate diffusional mobilities. Examples of such systems have been discussed in [16] and include some two-component and three-component Ni-base alloys. We note that references to some available experimental data on coarsening rates in those multicomponent alloy systems have also been provided in [16]. Moreover, our analytical findings not only offer further theoretical understanding to the simulation results documented in [16] but also conform with the experimentally observed coarsening rates. Thus, this study may shed new light on the microstructure coarsening process in realistic systems. In summary, since phase field models are widely used to model complex physical, material, or biological systems, our results indicate that diffusion mobility plays a very important role and it should be given special attention in relevant studies.

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