MULTISCALE MODEL. SIMUL.
Vol. 12, No. 4, pp. 1870–1889

COARSENING MECHANISM FOR SYSTEMS GOVERNED BY THE CAHN–HILLIARD EQUATION WITH DEGENERATE DIFFUSION MOBILITY*

SHIBIN DAI† AND QIANG DU‡

Abstract. We study a Cahn–Hilliard equation with a diffusion mobility that is degenerate in both phases and a double-well potential that is continuously differentiable. Using asymptotic analysis, we show that the interface separating the two phases does not move in the $t = O(1)$ or $O(\varepsilon^{-1})$ time scales, although in the latter regime there is a nontrivial porous medium diffusion process in both phases. Interface motion occurs in the $t = O(\varepsilon^{-2})$ time scale and is determined by quasi-stationary porous medium diffusion processes in both bulk phases, together with a surface diffusion process along the interface itself. In addition, in off-critical systems where one phase—the minor phase—occupies only a small fraction of the system and consists of many disjoint components, it is the quasi-stationary porous medium diffusion process that provides communications between the disjoint components and accounts for the occurrence of coarsening.

Key words. Cahn–Hilliard equation, degenerate diffusion mobility, asymptotic analysis, coarsening, motion of interfaces

AMS subject classifications. 35B40, 35Q92, 74N20, 82C26

DOI. 10.1137/140952387

1. Introduction.

The Cahn–Hilliard equation is a phenomenological diffuse-interface model for phase separation. It is written as

$$u_t = \nabla \cdot (M(u)\nabla\mu) \quad \text{for } x \in \Omega \subset \mathbb{R}^n,\ t \in [0,\infty), \tag{1.1}$$

$$\mu := -\varepsilon^2 \Delta u + F'(u). \tag{1.2}$$

Here Ω is an open subset of $\mathbb{R}^n$. Typically u represents the relative concentration of the two phases, and $F(u)$ is a double-well potential with two equal minima at $u^- < u^+$ which represent the relative concentrations of the two pure phases. When there is no ambiguity, we simply call $u^\pm$ the two pure phases for convenience. A widely used double-well potential with two equal wells is

$$F(u) = \frac{1}{4}(1-u^2)^2, \tag{1.3}$$

for which the two minima are $u^\pm = \pm 1$. The system (1.1)–(1.2) is often coupled with either the Neumann-type boundary condition

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \mu = 0$$

*Received by the editors January 13, 2014; accepted for publication (in revised form) October 21, 2014; published electronically December 16, 2014.
http://www.siam.org/journals/mms/12-4/95238.html
†Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003 (sdai@nmsu.edu). The research of this author was supported in part by US NSF grant DMS-1411438.
‡Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, and Department of Mathematics, Pennsylvania State University, University Park, PA 16802 (qd2125@columbia.edu). The research of this author was supported in part by US NSF grant DMS-1318586.

or a periodic boundary condition with Ω being the periodic cell. The resulting model dissipates the free energy, which is defined by

$$E(u) = \int_\Omega \left\{ \frac{\varepsilon^2}{2}|\nabla u|^2 + F(u) \right\} dx\,, \tag{1.4}$$

as long as the diffusion mobility $M(u)$ remains nonnegative, and exhibits a coarsening phenomenon. For numerical simulations, see, e.g., [2, 11, 12, 28, 33] and the references cited therein.

The dynamics of the system is influenced by the double-well potential F and the diffusion mobility M. For the case where M is a constant independent of u, and F is smooth in u, the migration of the sharp interface separating the two phases was shown to be described by the Mullins–Sekerka model, [1, 13, 27], for which the interface motion is determined by a diffusion process in two bulk phases.

The use of variable mobilities goes back to Cahn's work in [8]. The case where M is degenerate in both pure phases, e.g.,

$$M(u) = (u^+ - u)(u - u^-), \tag{1.5}$$

has been discussed in [10, 31] so that the Cahn–Hilliard equation describes interface motions determined by diffusions along the interface itself, with no diffusion in the bulk phases. It is shown in [9] by asymptotic analysis that this is true for the following two situations, in which the double-well potentials are different from smooth ones like (1.3).

(i). The first, in notation we use here, is when M is chosen as (1.5) and F is a double-barrier potential

$$F_{db}(u) = \begin{cases} \dfrac{1}{2}(u^+ - u)(u - u^-) & \text{if } u^- \le u \le u^+, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

The interface motion is indeed determined by surface diffusion. F_{db} does have two equal minima at $u^\pm$; hence $M(u)$ is degenerate in the two pure phases where $u = u^\pm$. However, F_{db} has singularities at $u^\pm$ since it not only has infinite barriers outside of $[u^-, u^+]$ but also has nonzero right derivative at u^- and nonzero left derivative at u^+.

(ii). The second situation needs special attention. In this case F is chosen as the following logarithmic potential:

$$\hat{F}(u) = \frac{\theta}{2}\Big((1+u)\ln(1+u) + (1-u)\ln(1-u) \Big) + \frac{1}{2}(1-u^2). \tag{1.7}$$

Here $\theta > 0$ is a small parameter representing temperature. $\hat{F}$ is defined only for $u \in [-1, 1]$, and

$$\lim_{u\to 1^-} \hat{F}'(u) = +\infty, \qquad \lim_{u\to(-1)^+} \hat{F}'(u) = -\infty.$$

The singularities of $\hat{F}$ at ± 1 effectively form an infinite barrier at ± 1. Hence $\hat{F}$ is usually coupled with a mobility

$$\hat{M}(u) = \begin{cases} 1 - u^2 & \text{if } u \in [-1, 1], \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

which is degenerate at ± 1. It is shown in [9] that when $\hat{F}$ is coupled with $\hat{M}$, if $\theta = O(\varepsilon^\alpha)$ for some appropriate power $\alpha > 0$, then formally in the limit $\varepsilon \to 0$ the interface motion is determined by surface diffusion (see also [19] for a corresponding result for systems of degenerate Cahn–Hilliard equations). A more detailed review can be found in [26].

Even though $\hat{F}$ formally approximates the double barrier potential F_{db} as $\theta \to 0$, these two situations are genuinely different. $\hat{F}$ does have two equal minima as long as θ is small enough, but the two minimizers of $\hat{F}$ are $\hat{u}^\pm = \pm(1 - \hat{\mathcal{R}}(\theta))$, and $\hat{\mathcal{R}}(\theta)$ is a small positive term that decays to 0 as $\theta \to 0$. Hence for fixed $\theta > 0$, the degeneracy of $\hat{M}(u)$ is not in the two pure phases $\hat{u}^\pm$. Indeed,

$$\hat{M}(\hat{u}^\pm) = 2\hat{\mathcal{R}}(\theta) - \hat{\mathcal{R}}(\theta)^2,$$

which is small but always positive. In this case, if we want the mobility to be degenerate in the two pure phases, then it would amount to choosing $M(u) = (1-\hat{\mathcal{R}}(\theta))^2 - u^2$.

Given the above differences, when choosing a smooth double-well potential such as (1.3), which has a simple form for numerical simulation and is hence widely used for phase-field modeling, together with a diffusion mobility that is degenerate in the two pure phases corresponding to the two minimizers $u^\pm$ of the double-well potential, it is natural to ask what different, if any, interfacial motion would be observed. In [10], Cahn and Taylor claimed that in this case the system does not allow pure phases, that is, regions where u equals exactly $u^\pm$, to exist. Consequently diffusion processes exist in bulk phases. They also mentioned in [10] that unpublished numerical simulations by W. Craig Carter support such a claim.

In [14], we studied a closely related problem, which is the Cahn–Hilliard equation with a smooth double-well potential, and a diffusion mobility $M(u) = u - u^-$. This M is degenerate in the negative pure phase $u = u^-$ and nondegenerate in the positive pure phase $u = u^+$. We find that in the slower $t = O(\varepsilon^{-2})$ time scale, there is a quasi-stationary porous medium diffusion process in the degenerate phase, which drives the interface separating the two phases to move. This result suggests that, if the diffusion mobility is degenerate in both pure phases, and if the double-well potential is smooth, in addition to surface diffusion, there should be a quasi-stationary porous medium diffusion process in both phases, and it should contribute to the geometric evolution of the interface. This provides additional motivation to the work presented in this paper.

We note that there have been a great deal of studies on the Cahn–Hilliard system in the literature which are beyond our scope here to comprehensively review. We only mentioned a few works closely related to our main focus in the above discussion.

1.1. Main result: Interface motion law. It is the purpose of this paper to explore the interface motion for the Cahn–Hilliard equation with a smooth double-well potential F, and a diffusion mobility that is degenerate in the two pure phases $u^\pm$ corresponding to the two minimizers of F. First, we notice that for the Cahn–Hilliard equation to be meaningful and actually dissipate the free energy E, the natural requirement is $M(u) \geq 0$ for all u. If $M(u) = (u^+ - u)(u - u^-)$, then the nonnegativity of M requires $u \in [u^-, u^+]$ for all time. This is not an issue when the energy has a double-barrier potential such as (1.6), since the barriers exclude the possibility for the values of u to actually go outside of the interval $[u^-, u^+]$. But for a double-well potential that is smooth at the two minimizers $u^\pm$, such as (1.3), from a purely energetic point of view, there is no mathematical reason to exclude perturbations that cause u to go outside of $[u^-, u^+]$.

In terms of physics of phase transitions, when the interface separating the two phases has zero mean curvature, there is no excessive surface energy, and hence the concentrations of each phase can be considered standard concentrations; in our language such phases are the pure phases. The Gibbs–Thomson effect indicates that, if the interface has nonzero mean curvature, then the curved interface has excessive surface energy, which affects the chemical potential in the two phases. Consequently the concentration of a phase inside small particles with high mean curvature is higher than that of the corresponding pure phase.

The Cahn–Hilliard equation (1.1)–(1.2) models phase separations of two phases, and the minimizers $u^{\pm}$ of the double-well potential F correspond to the relative concentration of the two phases. So if the Cahn–Hilliard equation accommodates the Gibbs–Thomson effect, then it has to allow u to be bigger than u^+ in convex droplets of positive phase and smaller than u^- in convex droplets of negative phase. For example, the Cahn–Hilliard equation with logarithmic potential $\hat{F}$ and degenerate mobility $\hat{M}$ accommodates the Gibbs–Thomson effect. Even though the solution u is always bounded in $[-1,1]$ due to the barriers of $\hat{F}$ at ± 1 and the cutoff of $\hat{M}$ outside of $[-1,1]$, since $u^{\pm} = \pm(1-\hat{\mathcal{R}}(\theta))$, the system allows u to be bigger than u^+ in some convex regions and smaller than u^- in some other small convex regions. In contrast, the Cahn–Hilliard equation with double-barrier potential F_{db} is not compatible with the Gibbs–Thomson effect due to the singularities of F_{db} at $u^{\pm}$.

So when the double-well potential is smooth at its minimizers $u^{\pm}$, and the mobility $M(u)$ is degenerate at $u^{\pm}$, in order to accommodate the Gibbs–Thomson effect, we have to allow u to take values outside of $[u^-,u^+]$, and consequently the mobility $M(u)$ has to be extended for values of u outside of $[u^-,u^+]$. To guarantee the nonnegativity of $M(u)$, in this paper we let $\tilde{M}(u) = (u^+-u)(u-u^-)$ and choose

$$M(u) = |\tilde{M}(u)| = |(u^+-u)(u-u^-)| \quad \text{for all } u. \tag{1.9}$$

This choice gives a mobility that has an order 1 degeneracy at $u = u^{\pm}$. Mathematically, degenerate diffusion mobilities can have other forms. For example, $M(u) = \tilde{M}(u)^2 = (u^+-u)^2(u-u^-)^2$ is smoother and has higher order degeneracy at $u = u^{\pm}$. In what follows, we often intentionally use $\tilde{M}(u)$ and $\tilde{M}'(u)$ rather than their more explicit forms so as to highlight the effect of specific forms of diffusion mobilities on the derivations.

Remark 1.1. In [16] it was proved that, given a constant $m \geq 1$, for

$$M(u) = \begin{cases} (1-u^2)^m & \text{if } u \in [-1,1], \\ 0 & \text{otherwise,} \end{cases}$$

and a sufficiently smooth potential in $[-1,1]$, there exists a solution, in a weak sense, of (1.1)–(1.2) that lies in $[-1,1]$ for all time, provided the initial value satisfies a finite energy condition that matches with the degeneracy of M. The cutoff of M outside of $[-1,1]$ is meaningful when combined with the logarithmic potential (1.7) or double-barrier potentials F satisfying $\|F\|_{C^2[-1,1]} < \infty$ and $F(u) = \infty$ for $|u| > 1$. After a simple rescaling, the results in [16] can be applied to a smooth double-well potential F with minima at $u^{\pm}$, coupled with a cutoff mobility

$$M_c(u) = \begin{cases} (u-u^-)^m(u^+-u)^m & \text{if } u \in [u^-,u^+], \\ 0 & \text{otherwise.} \end{cases}$$

It can then be shown that, as long as $\|F\|_{C^2[u^-,u^+]} < \infty$, there exists a weak solution u lying inside $[u^-,u^+]$ for all time. However, there is a lack of justification for such a

cutoff in the mobility. Indeed, the discussion above indicates that for a mathematical model to be compatible with the Gibbs–Thomson effect, it has to allow u to eventually go outside of $[u^-, u^+]$, even if the initial values of u lie in (u^-, u^+). In addition, if initially there is some region ω where $u \notin [u^-, u^+]$, then the cutoff in the mobility would prevent phase separation in ω, which is undesirable. In comparison, our choice (1.9) in principle does not have such restrictions on initial values of u, and the asymptotic analysis shows that the sharp interface limit derived here is fully compatible with the Gibbs–Thomson effect; see (1.14).

Remark 1.2. With our choice M as (1.9) and a smooth double-well potential such as (1.3), although a rigorous proof is still lacking, the existence of a solution u for (1.1)–(1.2) that goes outside of $[u^-, u^+]$, and hence bears the Gibbs–Thomson effect, is not in contradiction to the weak solution in [16], since for degenerate equations, weak solutions are usually not unique. For example, for the one-dimensional thin film equation $h_t + (|h|^n h_{xxx})_x = 0$, it is proven that there is a nonnegative weak solution [4]. But even if the initial value is positive, singularities may develop in finite time [3] in the sense that h reaches 0 in finite time, and there also exists a solution h that takes both positive and negative values [6]. While further theoretical exploration is needed, numerical simulations show that, if M is taken as $(u-u^-)(u^+-u)$ without the absolute value or cutoff outside of $[u^-, u^+]$, then even if the initial value of u lies in (u^-, u^+), within finite time u will go outside of $[u^-, u^+]$ and the system will become ill-posed. There have been claims that round-off errors might be to blame, and the mobility may be approximated by nondegenerate mobilities like $M^\varepsilon(u) = |(u-u^-)(u^+-u)| + \varepsilon^2$; see, for instance, [12] and references therein. Our formal result shows that the Gibbs–Thomson effect is a genuine feature of the model and not an artifact due to round-off errors.

Remark 1.3. There has also been a tendency in various earlier works to believe that regions away from the interface are pure phases. For example, based on this belief, in [22] it was argued that any interface with nonzero mean curvature is not in the asymptotic regime. There is, however, a misconceived argument there due to the nonuniqueness of solutions to a differential equation. Writing $u = u_0 + \varepsilon u_1 + u_2\varepsilon^2 + \dots$, it was noticed in [22] that $u = u^\pm$ solves the outer equation in all orders; hence $u_i = 0$ is taken for $i = 1, 2, \dots$. But what this argument missed is the fact that the first order outer equation has many solutions if one does not take into account the boundary condition on the interface. The correct value of u_1 is determined by the boundary condition on the interface, which is determined by the inner solution. When the interface has nonzero mean curvature, the boundary condition on the interface for the outer solution is the Gibbs–Thomson condition, which drives u_1 away from zero. Hence $u = u^\pm + \varepsilon u_1 + O(\varepsilon^2)$ is not exactly $u^\pm$ in the outer regions.

In the rest of the paper, to simplify notation we follow a conventional normalization so that the two minimizers of the double-well potential F are $u^\pm = \pm 1$. Hence

$$M(u) = |\tilde{M}(u)| \quad \text{and} \quad \tilde{M}(u) = 1 - u^2.$$

In addition, we assume that ± 1 are nondegenerate minimizers of F; that is, $F'(\pm 1) = 0$, $F''(\pm 1) > 0$.

The main tool used in the current work is asymptotic analysis, assuming that the solution u to (1.1)–(1.2) is differentiable up to whatever order necessary. In general, degenerate parabolic equations only allow generalized solutions (see [16, 32] for generalized bounded solutions for the Cahn–Hilliard equation with degenerate diffusion mobility) rather than classical ones, and generalized solutions are usually

not unique. Nevertheless, generalized solutions may still have enough regularity to make the results in this paper meaningful. Since we are only interested in the long time dynamics, we assume that the domain $\Omega \subset \mathbb{R}^n$ can be decomposed into three regions: Ω_+ in which $u \approx +1$, Ω_- in which $u \approx -1$, and $\Gamma_\varepsilon := \Omega \setminus (\Omega_+ \cup \Omega_-)$, which is a thin region around the level set $\Gamma := \{x \in \Omega : u(x) = 0\}$. Note that $\Omega_\pm$, Γ_ε, and Γ all depend on time. We assume Γ to be a collection of simple, closed, and sufficiently smooth surfaces in $\mathbb{R}^n$.

Since u depends on ε, Γ in fact also depends on ε. Formally, in the limit $\varepsilon \to 0$, Γ has a limit $\hat{\Gamma}$ and the transition region Γ_ε also shrinks to $\hat{\Gamma}$. Accordingly the outer and inner regions $\Omega_\pm$ converge respectively to two regions $\hat{\Omega}_\pm$ separated by $\hat{\Gamma}$. What we really want is to describe the motion of $\hat{\Gamma}$, but formally it is analogous to that of Γ. In the rest of the paper we do not differentiate $\hat{\Gamma}$ from Γ, and $\hat{\Omega}_\pm$ from $\Omega_\pm$.

Let $\mathbf{n}$ be the unit normal of Γ pointing toward Ω_+, $\{k_i, i = 1, \dots, n-1\}$ the principal curvatures of Γ, and $\kappa_0 = k_1 + \cdots + k_{n-1}$ the mean curvature. Under such an orientation, if Γ is convex, then its center of curvature lies in Ω_-, and the mean curvature is positive. We denote by Δ_s the Laplace–Beltrami operator on Γ and by $[\cdot]_-^+$ the jump of a function across Γ; that is, for any function f and any point $x \in \Gamma$,

$$[f(x)]_-^+ := f(x)^+ - f(x)^- = \lim_{h \to 0^+} f(x + h\mathbf{n}) - \lim_{h \to 0^+} f(x - h\mathbf{n}). \tag{1.10}$$

Let $U = U(z)$ represent the standard transition layer profile satisfying

$$U''(z) = F'(U(z)), \quad U(0) = 0, \quad \lim_{z \to \pm\infty} U(z) = \pm 1. \tag{1.11}$$

For the double-well potential F defined by (1.3), $U(z) = \tanh(z/\sqrt{2})$. The following constants are defined by U:

$$S = \frac{1}{2} \int_{-\infty}^{\infty} U'(z)^2 \, dz, \quad \sigma = \frac{S}{2} \int_{-\infty}^{\infty} (1 - U^2) \, dz. \tag{1.12}$$

Our main result is the following statement about the quasi-equilibrium evolution.

PRINCIPLE RESULT 1.4. *If $M(u) = |\tilde{M}(u)| = |1 - u^2|$ and F is a smooth double-well potential, the nontrivial interface motion under the Cahn–Hilliard equation* (1.1)–(1.2) *occurs in the time scale $t_2 = \varepsilon^2 t$, or $t = O(\varepsilon^{-2})$, and the normal velocity $V_{\mathbf{n}}$ of the interface is determined by the following free boundary problem:*

$$\nabla \cdot (|\mu_1| \nabla \mu_1) = 0 \qquad \text{in } \Omega_\pm, \tag{1.13}$$

$$\mu_1 = -S\kappa_0 \qquad \text{on } \Gamma, \tag{1.14}$$

$$V_{\mathbf{n}} = \sigma \Delta_s \kappa_0 - \left[\frac{|\tilde{M}'(u_0)\mu_1|}{2F''(u_0)} \partial_{\mathbf{n}} \mu_1 \right]_-^+ \qquad \text{on } \Gamma. \tag{1.15}$$

Here $u_0 = \pm 1$ in $\Omega_\pm$, respectively.

Remark 1.5. In (1.15), the effect of the diffusion field on the interface motion is incorporated as a jump of fluxes. Even though we only work out the details for the specific diffusion mobility $M(u) = |\tilde{M}'(u)| = |1 - u^2|$ here, it is expected that similar observations hold for generic diffusion mobilities. Specifically, if $\tilde{M}(u)$ is of higher order degeneracy in both Ω_+ and Ω_- so that $\tilde{M}'(u_0) = 0$, then the diffusion process in the corresponding bulk phase will not influence the interface motion in the t_2 time scale; its influence can only be in a slower time scale. In contrast, surface diffusion $\Delta_s \kappa_0$ always occurs in the t_2 time scale. Rigorous analysis of the existence and properties of a solution to (1.13)–(1.15) is beyond the scope of the paper. For the study of surface diffusion itself, we refer the reader to [17, 18].

1.2. Coarsening mechanism. One key feature of the Cahn–Hilliard equation (1.1)–(1.2) is the dissipation of the free energy E defined by (1.4), which leads to the coarsening phenomenon. Since the counterpart of E in a sharp-interface model is $|\Gamma|$, the total area of the interface Γ, it is of interest to study the evolution of $|\Gamma|$ under (1.13)–(1.15). To this end, even if Γ consists of disjoint pieces, the porous medium diffusion processes in $\Omega_\pm$ provide communications between these pieces and make coarsening possible. We have

$$\frac{d|\Gamma|}{dt_2} = \int_\Gamma \kappa_0 V_{\mathbf{n}}\, ds = \sigma \int_\Gamma \kappa_0 \Delta_s \kappa_0\, ds + \frac{1}{S}\int_\Gamma \mu_1 \left[\mathbf{n}\cdot\left(\frac{|\tilde{M}'(u_0)\mu_1|}{2F''(u_0)}\nabla\mu_1\right)\right]_-^+ ds$$

$$= -\sigma \int_\Gamma |\nabla_s \kappa_0|^2\, ds - \frac{1}{S}\int_{\Omega_+\cup\Omega_-} \frac{|\tilde{M}'(u_0)\mu_1|}{2F''(u_0)}|\nabla\mu_1|^2\, dx \le 0. \tag{1.16}$$

By (1.16), we expect the interfacial area to be always decreasing, until the system settles into an equilibrium state, in which each piece of Γ has constant mean curvature, and μ_1 is constant in each connected component of Ω_+ and Ω_-.

A specifically interesting situation is the off-critical case when one phase, the minor phase, occupies only a small fraction of the system. In [7], using a heuristic argument, Bray and Emmott derived a mean field model for the Cahn–Hilliard equation with degenerate diffusion mobility in the dilute limit of the off-critical case when the minor phase consists of many disjoint small spherical particles. In this case, the surface diffusion is missing since the interface is a collection of spheres. Under this simplified geometric configuration, their heuristic argument suggests that the dynamics is determined by a diffusion process, which they called subdiffusion, in the major phase. Our sharp-interface model (1.13)–(1.15) goes beyond the results in [7] in that we capture the surface diffusion effect together with the quasi-stationary porous medium diffusion process. In addition, we can recover the results in [7] in the off-critical case. One key ingredient is the following lemma, whose proof is given in the appendix.

LEMMA 1.6. *Let ω be an open, bounded, and connected subset of $\mathbb{R}^n$ with smooth boundary. Then for any $g \in C^2(\omega)\cap C(\bar{\omega})$ with $g \ge 0$ on $\partial\omega$, the equation*

$$\nabla\cdot(|\mu|\nabla\mu) = 0 \qquad \text{in } \omega, \tag{1.17}$$

$$\mu = g \ge 0 \qquad \text{on } \partial\omega \tag{1.18}$$

has a unique solution μ in $C^2(\omega)\cap C(\bar{\omega})$ and $\mu \ge 0$ in ω. Furthermore, if $g > 0$ on some point $x \in \partial\omega$, then $\mu > 0$ in ω.

Depending on whether the minor phase is Ω_+ or Ω_-, our analysis is slightly different.

Case 1. *Ω_+ is the minor phase.* Let $\Omega_+ = \cup_{i=1}^N \omega_+^i$, where each ω_+^i is a convex set, the interface $\Gamma = \cup_{i=1}^N \partial\omega_+^i$. Since the normal $\mathbf{n}$ of Γ points toward Ω_+, the mean curvature κ_0 is negative on each $\partial\omega_+^i$. By (1.13)–(1.14) we have

$$\nabla\cdot(|\mu_1|\nabla\mu_1) = 0 \qquad \text{in } \omega_+^i, \tag{1.19}$$

$$\mu_1 = -S\kappa_0 > 0 \qquad \text{on } \partial\omega_+^i. \tag{1.20}$$

By Lemma 1.6, the unique C^2 solution μ_1 for (1.19)–(1.20) is positive in each ω_+^i.

Now $\Omega_- = \Omega\setminus\cup_{i=1}^N \omega_+^i$, and we need to solve (1.13)–(1.14) to find μ_1 in Ω_-. Since $\partial\Omega_- = \cup_{i=1}^N \partial\omega_+^i \cup \partial\Omega$ and μ_1 satisfies either periodic or Neumann boundary conditions

on $\partial\Omega$, a slight variance of Lemma 1.6 shows that $\mu_1 > 0$ in Ω_-. Since our solution $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$ in $\Omega_\pm$ (see section 2 for details), and $u_1 = \mu_1/F''(u_0) > 0$, we have $u > u_0 = \pm 1$ in $\Omega_\pm$, respectively. This indicates that the curvature effect through the Gibbs–Thomson condition (1.14) drives the values of u in each ω_+^i bigger than 1, and in Ω_- bigger than -1.

If every ω_+^i is a sphere of radius r_i, then $\kappa_0 = -(n-1)/r_i$ on $\partial\omega_+^i$, and $\mu_1 = (n-1)S/r_i$ is constant in each ω_+^i. Then there is no surface diffusion since $\Delta_s\kappa_0 = 0$ on Γ, and there is no flux from ω_+^i. The coarsening is driven by the diffusion field in Ω_-. This recovers the result in [7].

Case 2. Ω_- *is the minor phase.* In this case, $\Omega_- = \cup_{i=1}^N \omega_-^i$, each ω_-^i is a convex set. The interface $\Gamma = \cup_{i=1}^N \partial\omega_-^i$. On $\partial\omega_-^i$, the mean curvature κ_0 is positive since the normal $\mathbf{n}$ points toward Ω_+. In each ω_-^i,

$$\nabla \cdot (|\mu_1|\nabla\mu_1) = 0 \qquad \text{in } \omega_-^i, \tag{1.21}$$

$$\mu_1 = -S\kappa_0 < 0 \qquad \text{on } \partial\omega_-^i. \tag{1.22}$$

By Lemma 1.6, it has a unique negative solution $\mu_1 < 0$ in each ω_-^i. Similarly, $\mu_1 < 0$ in Ω_+. The solution $u = u_0 + \varepsilon u_1 + O(\varepsilon^2) < u_0 = \pm 1$ in $\Omega_\pm$, respectively. Hence the curvature effect drives the values of u smaller than -1 in Ω_- and smaller than 1 in Ω_+. Similar to Case 1, if all ω_-^i are spherical, then coarsening is driven by the diffusion field in Ω_+.

In the off-critical situation, the quasi-stationary porous medium diffusion process in the major phase is the mechanism for the disjoint components of the minor phase to communicate and hence for the occurrence of coarsening. In the critical situation when each phase occupies half of the system, the interface is essentially connected. This makes it very hard to justify the presence of the quasi-stationary porous medium diffusion process, since surface diffusion itself can induce coarsening. Our asymptotic analysis indicates that it is the combination of this diffusion process and surface diffusion that accounts for coarsening. We again associate such findings to numerical simulations of the Cahn–Hilliard equation with degenerate mobility; see, e.g., [12, 33].

1.3. Coarsening rate. Coarsening is featured by the growth of a characteristic length scale $l(t)$. Under the assumption that the system morphology is self-similar, in general $l(t)$ grows like a power law of time: $l(t) \sim ct^\alpha$. For the Cahn–Hilliard equation with constant mobility, $\alpha = 1/3$. When the mobility is $|1-u^2|$, which is degenerate in both phases, numerical simulations are mainly about the critical case when each phase occupies half of the system, and simulations indicate that $\alpha = 1/4$. Since our interface motion law (1.13)–(1.15) is independent of volume fraction of the phases, we will show that this 1/4 power law is a consequence of scaling invariance of (1.13)–(1.15).

Take X, T, and Ψ as typical length, time, and chemical potential scales, and rescale:

$$x = X\hat{x}, \quad t = T\hat{t}, \quad \mu_1 = \Psi\hat{\mu}_1. \tag{1.23}$$

Then

$$V_{\mathbf{n}} = \hat{V}_{\mathbf{n}}\frac{X}{T}, \quad \kappa_0 = \frac{1}{X}\hat{\kappa}_0, \quad \nabla = \frac{1}{X}\hat{\nabla}, \quad \Delta_s = \frac{1}{X^2}\hat{\Delta}_s. \tag{1.24}$$

Plugging (1.23) and (1.24) into (1.13)–(1.15), we have

$$\hat{\nabla}\cdot(|\hat{\mu}_1|\hat{\nabla}\hat{\mu}_1)=0 \qquad \text{in } \Omega_{\pm}, \tag{1.25}$$

$$\Psi\hat{\mu}_1=-\frac{S}{X}\hat{\kappa}_0 \qquad \text{on } \Gamma, \tag{1.26}$$

$$\frac{X}{T}\hat{V}_{\mathbf{n}}=\frac{1}{X^3}\sigma\hat{\Delta}_s\hat{\kappa}_0-\frac{\Psi^2}{X}\left[\frac{|\tilde{M}'(u_0)\hat{\mu}_1|}{2F''(u_0)}\partial_{\mathbf{n}}\hat{\mu}_1\right]_-^+ \qquad \text{on } \Gamma. \tag{1.27}$$

By (1.26), a natural choice of Ψ is X^{-1}. Then, if we choose $T=X^4$, (1.25)–(1.27) restore to the form of (1.13)–(1.15). This spatiotemporal relation $T=X^4$ indicates a growth rate of the characteristic length $l(t)\sim ct^{1/4}$. While the characteristic exponent $1/4$ is the same as that predicted by a stand-alone surface diffusion model, (1.13)–(1.15) and the scaling analysis demonstrate that the bulk nonlinear diffusion and the surface diffusion both contribute to the coarsening process.

The structure of this paper is as follows. In section 2 we establish the inner local frame and the inner/outer expansions. In section 3 we study the inner structure of the transition region. In section 4 we derive the dynamics in the $t_1=\varepsilon t$ time scale, which is a porous medium equation. In section 5 we derive the dynamics in the $t_2=\varepsilon^2 t$ time scale, which gives (1.13)–(1.15). Finally, in section 6 we discuss the merit of our results and applications.

2. Inner local frame and inner/outer expansions. To study the inner structure in the neighborhood Γ_ε of the interface Γ, we need to use a local coordinate system. This is a standard procedure, although different authors take slightly different approaches under various assumptions; see, e.g., [9, 14, 15, 20, 27]. We assume Γ is a collection of simple, closed, smooth surfaces in $\mathbb{R}^n$, which means that there is a parametric representation, at least locally,

$$\Gamma=\{\phi(s): s=(s_1,\dots,s_{n-1})\}. \tag{2.1}$$

The parameter s can be chosen so that $\{\mathbf{T}^i:=\frac{\partial\phi}{\partial s_i}, i=1,\dots,n-1\}$ form an orthonormal basis for the tangent plane of Γ at $\phi(s)$, and $\frac{\partial\mathbf{n}}{\partial s_i}=k_i\mathbf{T}^i, i=1,\dots,n-1$. Here $\mathbf{n}$ is the normal of Γ pointing toward Ω_+, and $\{k_i, i=1,\dots,n-1\}$ are the principal curvatures of Γ. We assume further that any point x in Γ_ε can be uniquely represented by its orthogonal projection $\phi(s)$ on Γ and its signed distance $\rho(x)=(x-\phi(s))\cdot\mathbf{n}(s)$ to Γ. By defining $r=\rho(x)$, we have a local coordinate system (s,r), whose relation to the Cartesian coordinates x is

$$x=\phi(s)+r\mathbf{n}(s). \tag{2.2}$$

The following lemma summarizes some results we will use later; for a proof, see [14].

LEMMA 2.1 (see [14]).

(i) $-\frac{\partial\rho}{\partial t}$ *is the normal velocity of* Γ*, positive when moving in the direction of* $\mathbf{n}$.

(ii) $\nabla_x s_i=\frac{1}{1+rk_i}\mathbf{T}^i, \quad \Delta_x s_i=-\frac{r}{(1+rk_i)^3}\frac{\partial k_i}{\partial s_i}, \quad i=1,\dots,n-1.$

(iii) $\nabla_x r=\mathbf{n}, \quad \Delta_x r=\sum_{j=1}^{n-1}\frac{k_j}{1+rk_j}$.

(iv) *In the local coordinates* (s,r)*, the Cartesian Laplacian* Δ_x *has the following*

expression:

$$\Delta_x = \sum_{j=1}^{n-1} \frac{1}{(1+rk_j)^2} \frac{\partial^2}{\partial s_j^2} + \left(\sum_{j=1}^{n-1} \frac{k_j}{1+rk_j} \right) \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \sum_{j=1}^{n-1} \left(-r \frac{\partial k_j}{\partial s_j} \frac{1}{(1+rk_j)^3} \right) \frac{\partial}{\partial s_j}. \tag{2.3}$$

Rescaling the normal coordinate by $z = r/\varepsilon$ and expanding (2.3) in powers of ε, we have

$$\Delta_x = \varepsilon^{-2}\partial_{zz} + \varepsilon^{-1}\kappa_0\partial_z + \Delta_0 + \varepsilon\Delta_1 + O(\varepsilon^2), \tag{2.4}$$

where

$$\Delta_0 := \Delta_s + z\kappa_1\partial_z, \quad \kappa_0 := \sum_{j=1}^{n-1} k_j, \quad \kappa_1 := -\sum_{j=1}^{n-1} k_j^2. \tag{2.5}$$

The operator Δ_1 characterizes the $O(\varepsilon)$ term in the expansion, but its explicit form is immaterial. κ_0 is the mean curvature of Γ. When $n = 3$, $\kappa_1 = 2k_1k_2 - (k_1+k_2)^2 = 2K - \kappa_0^2$, where K is the Gaussian curvature of Γ.

2.1. Inner expansion. In the transition region Γ_ε, let $u(x,t) = \tilde{u}(s,z,t)$ and $\mu(x,t) = \tilde{\mu}(s,z,t)$ be expanded as

$$u(x,t) = \tilde{u}_0(s,z,t) + \varepsilon\tilde{u}_1(s,z,t) + \varepsilon^2\tilde{u}_2(s,z,t) + \varepsilon^3\tilde{u}_3(s,z,t) + \dots,$$
$$\mu(x,t) = \tilde{\mu}_0(s,z,t) + \varepsilon\tilde{\mu}_1(s,z,t) + \varepsilon^2\tilde{\mu}_2(s,z,t) + \varepsilon^3\tilde{\mu}_3(s,z,t) + \dots.$$

Then by (1.2) and (2.4), we have

$$\tilde{\mu}_0 = -\tilde{u}_{0zz} + F'(\tilde{u}_0), \tag{2.6}$$
$$\tilde{\mu}_1 = -\tilde{u}_{1zz} - \kappa_0\tilde{u}_{0z} + F''(\tilde{u}_0)\tilde{u}_1, \tag{2.7}$$
$$\tilde{\mu}_2 = -\tilde{u}_{2zz} - \kappa_0\tilde{u}_{1z} - \Delta_0\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}F'''(\tilde{u}_0)\tilde{u}_1^2, \tag{2.8}$$
$$\tilde{\mu}_3 = -\tilde{u}_{3zz} - \kappa_0\tilde{u}_{2z} - \Delta_0\tilde{u}_1 - \Delta_1\tilde{u}_0 + F''(\tilde{u}_0)\tilde{u}_3 + F'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}F^{(4)}(\tilde{u}_0)\tilde{u}_1^3. \tag{2.9}$$

The diffusion mobility $M(u) = |\tilde{M}(u)| = |1 - \tilde{u}^2|$ can be expanded as

$$M(u) = |1 - \tilde{u}_0^2 + \varepsilon\tilde{M}'(\tilde{u}_0)\tilde{u}_1 + \varepsilon^2(\tilde{M}(\tilde{u}_0)\tilde{u}_2 - \tilde{u}_1^2) + O(\varepsilon^3)|. \tag{2.10}$$

Since we are interested in the quasi-equilibrium long time behavior of the system, we will assume that the leading order transition profile $\tilde{u}_0$ reaches its equilibrium state, which is the standard transition profile $U(z)$ determined by

$$-U_{zz} + F'(U) = 0,$$
$$U(0) = 0, \quad U(z) \to \pm 1 \quad \text{as } z \to \pm\infty.$$

One feature of U is, since $F'(\pm 1) = 0$ and $F''(\pm 1) > 0$, U approaches exponentially to $u^\pm = \pm 1$ as $z \to \pm\infty$. This feature is critical for our asymptotic analysis, since it

provides a separation of orders in $|1-U^2|$. Assuming that $\tilde{u}_0 \sim \pm(1-e^{-|z|/\beta})$ for $z \to \pm\infty$, then by taking $\eta = \beta \ln \frac{1}{\varepsilon}$, we have

$$|1-\tilde{u}_0^2| \le \begin{cases} O(\varepsilon) & \text{if } |z| \ge \eta, \\ O(\varepsilon^2) & \text{if } |z| \ge 2\eta, \\ O(\varepsilon^3) & \text{if } |z| \ge 3\eta, \\ O(\varepsilon^4) & \text{if } |z| \ge 4\eta. \end{cases} \tag{2.11}$$

The separation of orders described in (2.11) is a nonstandard technique crucial to handle the degeneracy of the diffusion mobility. We first used such a technique in [14] to study the Cahn–Hilliard equation with a one-sided degenerate diffusion mobility.

Because of (2.11), we divide $(-\infty, +\infty)$ into subsets:

$$Z_0 := \{z : |z| < \eta\}, \quad Z_1 := \{z : \eta \le |z| < 2\eta\}, \quad Z_2 := \{z : 2\eta \le |z| < 3\eta\},$$
$$Z_3 := \{z : 3\eta \le |z| < 4\eta\}, \quad Z_4 := \{z : |z| \ge 4\eta\}.$$

Letting $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ be the characteristic functions of these sets, we have the following expansion of $1-\tilde{u}_0^2$:

$$\begin{aligned} 1-\tilde{u}_0^2 = {} & (1-\tilde{u}_0^2)\chi_0 + \varepsilon(1-\tilde{u}_0^2)\chi_1\varepsilon^{-1} + \varepsilon^2(1-\tilde{u}_0^2)\chi_2\varepsilon^{-2} + \varepsilon^3(1-\tilde{u}_0^2)\chi_3\varepsilon^{-3} \\ & + \varepsilon^4(1-\tilde{u}_0^2)\chi_4\varepsilon^{-4}. \end{aligned} \tag{2.12}$$

The first four terms on the right-hand side are, respectively, of orders $1, \varepsilon, \varepsilon^2, \varepsilon^3$, and the last one is a residual term of order ε^4 and higher.

Since $\tilde{u}_{0z}$ decays exponentially to 0 as $z \to \pm\infty$ at the same rate as $\tilde{u}_0$ approaches exponentially to ± 1, we have a similar expansion for $\tilde{u}_{0z}$:

$$\tilde{u}_{0z} = \chi_0\tilde{u}_{0z} + \varepsilon\chi_1\varepsilon^{-1}\tilde{u}_{0z} + \varepsilon^2\chi_2\varepsilon^{-2}\tilde{u}_{0z} + \varepsilon^3\chi_3\varepsilon^{-3}\tilde{u}_{0z} + \varepsilon^4\chi_4\varepsilon^{-4}\tilde{u}_{0z}. \tag{2.13}$$

Define $\Phi(u) := \operatorname{sgn}(1-u^2)$ such that

$$\Phi(u) = \begin{cases} 1 & \text{if } 1-u^2 > 0, \\ -1 & \text{if } 1-u^2 < 0, \\ 0 & \text{if } 1-u^2 = 0. \end{cases} \tag{2.14}$$

Then $M(u) = \Phi(\tilde{u})\tilde{M}(\tilde{u}) = \Phi(\tilde{u})(1-\tilde{u}^2)$, $M'(u) = \tilde{M}'(u)\Phi(\tilde{u}) = -2\tilde{u}\Phi(\tilde{u})$, and

$$\begin{aligned} \nabla_x \cdot (M(u)\nabla_x\mu) &= M(u)\Delta_x\mu + \nabla_x M(u)\cdot\nabla_x\mu \\ &= (1-\tilde{u}^2)\Phi(\tilde{u})\left(\varepsilon^{-2}\tilde{\mu}_{zz} + \varepsilon^{-1}\kappa_0\tilde{\mu}_z + \Delta_0\tilde{\mu} + \varepsilon\Delta_1\tilde{\mu} + O(\varepsilon^2)\right) \\ &\quad + \tilde{M}'(\tilde{u})\Phi(\tilde{u})\left(\sum_{j=1}^{n-1}\frac{\partial\tilde{u}}{\partial s_j}\frac{\partial\tilde{\mu}}{\partial s_j}\frac{1}{(1+\varepsilon z k_j)^2} + \varepsilon^{-2}\tilde{u}_z\tilde{\mu}_z\right). \end{aligned} \tag{2.15}$$

Plugging (2.12) and (2.13) into (2.15), we have

$$\nabla_x \cdot (M(u)\nabla_x\mu) = \varepsilon^{-2}P_{-2} + \varepsilon^{-1}P_{-1} + P_0 + \varepsilon P_1 + O(\varepsilon^2), \tag{2.16}$$

where

$$
\begin{aligned}
P_{-2} &= \Phi(\tilde{u})\Big\{\chi_0(1-\tilde{u}_0^2)\tilde{\mu}_{0zz} + \tilde{M}'(\tilde{u}_0)\chi_0\tilde{u}_{0z}\tilde{\mu}_{0z}\Big\},\\
P_{-1} &= \Phi(\tilde{u})\Big\{\chi_0(1-\tilde{u}_0^2)(\kappa_0\tilde{\mu}_{0z}+\tilde{\mu}_{1zz}) + \Big(\varepsilon^{-1}\chi_1(1-\tilde{u}_0^2) + \tilde{M}'(\tilde{u}_0)\tilde{u}_1\Big)\tilde{\mu}_{0zz}\\
&\quad + \tilde{M}'(\tilde{u}_0)\Big(\chi_0\tilde{u}_{0z}\tilde{\mu}_{1z} + (\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{0z}\Big) - 2\tilde{u}_1(\chi_0\tilde{u}_{0z}\tilde{\mu}_{0z})\Big\},\\
P_0 &= \Phi(\tilde{u})\Big\{\chi_0(1-\tilde{u}_0^2)\Big(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}\Big) + \Big(\varepsilon^{-1}\chi_1(1-\tilde{u}_0^2) + \tilde{M}'(\tilde{u}_0)\tilde{u}_1\Big)\\
&\quad \cdot(\kappa_0\tilde{\mu}_{0z}+\tilde{\mu}_{1zz}) + \Big(\varepsilon^{-2}\chi_2(1-\tilde{u}_0^2) - \tilde{u}_1^2 + \tilde{M}'(\tilde{u}_0)\tilde{u}_2\Big)\tilde{\mu}_{0zz}\\
&\quad + \tilde{M}'(\tilde{u}_0)\cdot\Big(\sum_{j=1}^{n-1}\frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} + \chi_0\tilde{u}_{0z}\tilde{\mu}_{2z} + (\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{1z}\\
&\quad + (\varepsilon^{-2}\chi_2\tilde{u}_{0z}+\tilde{u}_{2z})\tilde{\mu}_{0z}\Big) + \tilde{M}'(\tilde{u}_1)\Big(\chi_0\tilde{u}_{0z}\tilde{\mu}_{1z} + (\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{0z}\Big)\\
&\quad + \tilde{M}'(\tilde{u}_2)\chi_0\tilde{u}_{0z}\tilde{\mu}_{0z}\Big\},\\
P_1 &= \Phi(\tilde{u})\Big\{\chi_0(1-\tilde{u}_0^2)\cdot\Big(\Delta_1\tilde{\mu}_0 + \Delta_0\tilde{\mu}_1 + \kappa_0\tilde{\mu}_{2z} + \tilde{\mu}_{3zz}\Big) + \Big(\varepsilon^{-1}\chi_1(1-\tilde{u}_0^2)\\
&\quad + \tilde{M}'(\tilde{u}_0)\tilde{u}_1\Big)\Big(\Delta_0\tilde{\mu}_0 + \kappa_0\tilde{\mu}_{1z} + \tilde{\mu}_{2zz}\Big) + \Big(\varepsilon^{-2}\chi_2(1-\tilde{u}_0^2) - \tilde{u}_1^2 + \tilde{M}'(\tilde{u}_0)\tilde{u}_2\Big)\\
&\quad \cdot(\kappa_0\tilde{\mu}_{0z}+\tilde{\mu}_{1zz}) + \Big(\varepsilon^{-3}\chi_3(1-\tilde{u}_0^2) + \tilde{M}'(\tilde{u}_0)\tilde{u}_3 + \tilde{M}'(\tilde{u}_1)\tilde{u}_2\Big)\tilde{\mu}_{0zz}\\
&\quad + \tilde{M}'(\tilde{u}_0)\Big(\sum_{j=1}^{n-1}\frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_1}{\partial s_j} + \sum_{j=1}^{n-1}\frac{\partial\tilde{u}_1}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} - 2z\sum_{j=1}^{n-1}k_j\frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} + \chi_0\tilde{u}_{0z}\tilde{\mu}_{3z}\\
&\quad + (\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{2z} + (\varepsilon^{-2}\chi_2\tilde{u}_{0z}+\tilde{u}_{2z})\tilde{\mu}_{1z} + (\varepsilon^{-3}\chi_3\tilde{u}_{0z}+\tilde{u}_{3z})\tilde{\mu}_{0z}\Big)\\
&\quad + \tilde{M}'(\tilde{u}_1)\Big(\sum_{j=1}^{n-1}\frac{\partial\tilde{u}_0}{\partial s_j}\frac{\partial\tilde{\mu}_0}{\partial s_j} + \chi_0\tilde{u}_{0z}\tilde{\mu}_{2z} + (\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{1z}\\
&\quad + (\varepsilon^{-2}\chi_2\tilde{u}_{0z}+\tilde{u}_{2z})\tilde{\mu}_{0z}\Big) + \tilde{M}'(\tilde{u}_2)\Big((\varepsilon^{-1}\chi_1\tilde{u}_{0z}+\tilde{u}_{1z})\tilde{\mu}_{0z} + \chi_0\tilde{u}_{0z}\tilde{\mu}_{1z}\Big)\\
&\quad + \tilde{M}'(\tilde{u}_3)\chi_0\tilde{u}_{0z}\tilde{\mu}_{0z}\Big\}.
\end{aligned}
$$

2.2. Outer expansion. In the bulk phases $\Omega_\pm$, $u \approx u_0 := \mathbf{1}_{\Omega_+} - \mathbf{1}_{\Omega_-}$. So away from Γ_ε, we have the following outer expansion:

$$u(x,t) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots, \quad \mu(x,t) = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots.$$

By (1.2), we obtain

$$\mu_0 = F'(u_0) = 0, \quad \mu_1 = F''(u_0)u_1, \quad \mu_2 = F''(u_0)u_2 + \frac{1}{2}F'''(u_0)u_1^2 - \Delta u_0.$$

If $u_1 \neq 0$, then

$$M(u) = |1 - u^2| = |1 - (u_0 + \varepsilon u_1 + O(\varepsilon^2))^2| = |\tilde{M}'(u_0)u_1|\varepsilon + O(\varepsilon^2).$$

If $u_1 = 0$, we have $M(u) = O(\varepsilon^2)$, which can still be written as

$$M(u) = |\tilde{M}'(u_0)u_1|\varepsilon + O(\varepsilon^2).$$

Hence we have

$$\nabla \cdot \big(M(u)\nabla\mu\big) = \varepsilon^2 \nabla \cdot \Big(|\tilde{M}'(u_0)u_1|\nabla\mu_1\Big) + O(\varepsilon^3). \tag{2.17}$$

The following match condition connects the inner and outer expansions [27]. For fixed $x \in \Gamma$, we require

$$(\mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots)(x + \varepsilon z\mathbf{n}, t) \approx (\tilde{\mu}_0 + \varepsilon\tilde{\mu}_1 + \varepsilon^2\tilde{\mu}_2 + \dots)(s, z, t) \tag{2.18}$$

when εz is between $O(\varepsilon)$ and $o(1)$. Writing $\partial_{\mathbf{n}}$ as the directional derivative in the $\mathbf{n}$ direction, and $f^{\pm}(x) := \lim_{h\to 0^+} f(x \pm h\mathbf{n})$ for any function f and $x \in \Gamma$, Taylor expansions of the left-hand side require that

$$\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_0, \tag{2.19}$$

$$\mu_1^{\pm} + z\partial_{\mathbf{n}}\mu_0^{\pm} = \tilde{\mu}_1 + o(1) \quad \text{as } z \to \pm\infty, \tag{2.20}$$

$$\mu_2^{\pm} + z\partial_{\mathbf{n}}\mu_1^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_0^{\pm} = \tilde{\mu}_2 + o(1) \quad \text{as } z \to \pm\infty. \tag{2.21}$$

Similar match conditions for u can also be obtained.

3. Inner structure of the transition region. In this section we consider the inner structure of the transition region when the outer structure is as previously described. We summarize the results in the following claim, whose validity is verified by analyzing the dynamics in different time scales.

CLAIM 3.1. *The inner structure of the transition region can be described as follows.*

(a) *The leading order transition profile $\tilde{u}_0 = U(z)$ is equilibrated in the $t = O(\varepsilon)$ time scale, and it satisfies* (1.11). *Consequently, $\tilde{\mu}_0 = 0$.*

(b) *The first order correction $\tilde{\mu}_1$ is equilibrated in the $t = O(1)$ time scale, in which the interface Γ does not move. In addition, the equilibrium state of $\tilde{\mu}_1$ satisfies the Gibbs–Thomson condition $\tilde{\mu}_1 = -S\kappa_0$.*

3.1. The fast time scales $T_2 = t/\varepsilon^2$ and $T_1 = t/\varepsilon$: Equilibration of $\tilde{u}_0$. We start with a fast time scale $T_2 = t/\varepsilon^2$. For the outer solution, by (1.1) and (2.17), matching the terms for every order in ε, we have

$$\partial_{T_2}u_0 = \partial_{T_2}u_1 = \partial_{T_2}u_2 = \partial_{T_2}u_3 = 0.$$

In the T_2 time scale, nothing happens to the outer solution up to order $O(\varepsilon^3)$.

For the inner solution, we have

$$u_t = \varepsilon^{-2}\tilde{u}_{T_2} + \varepsilon^{-2}\nabla_s \tilde{u}\cdot\frac{\partial s}{\partial T_2} + \varepsilon^{-3}\frac{\partial \rho}{\partial T_2}\tilde{u}_z = \varepsilon^{-3}\frac{\partial \rho}{\partial T_2}\tilde{u}_{0z}\chi_0 + O(\varepsilon^{-2}). \tag{3.1}$$

Matching (3.1) and (2.16), the ε^{-3} term gives

$$\frac{\partial \rho}{\partial T_2}\tilde{u}_{0z}\chi_0 = 0.$$

Since $\tilde{u}_0$ is not constant in z, we have

$$\frac{\partial \rho}{\partial T_2} = 0. \tag{3.2}$$

So the normal velocity of the interface is zero in the leading order in the T_2 time scale.

Next we study the $T_1 = t/\varepsilon$ time scale. The outer solution satisfies

$$\partial_{T_1} u_0 = \partial_{T_1} u_1 = \partial_{T_1} u_2 = 0.$$

The inner solution gives $u_t = \varepsilon^{-2}\frac{\partial \rho}{\partial T_1}\tilde{u}_{0z}\chi_0 + O(\varepsilon^{-1})$. Comparing with (2.16), we have the following possibly nontrivial equation:

$$\frac{\partial \rho}{\partial T_1}\tilde{u}_{0z}\chi_0 = \chi_0\Phi(\tilde{u}_0)\frac{\partial}{\partial z}\bigg((1-\tilde{u}_0^2)\tilde{\mu}_{0z}\bigg). \tag{3.3}$$

Since we are only interested in the quasi-equilibrium dynamics in the long time scale, we will only consider the equilibrium state of (3.3), which needs to satisfy

$$\frac{\partial}{\partial z}\left((1-\tilde{u}_0^2)\tilde{\mu}_{0z}\right) = 0 \quad \text{for } z \in (-\eta, \eta) \tag{3.4}$$

with the following boundary conditions at $\pm\infty$:

$$\tilde{u}_0 \to u_0^{\pm} = \pm 1 \text{ exponentially as } z \to \pm\infty.$$

The position of the interface is determined by the condition $\tilde{u}_0(0) = 0$. The standard transition profile U defined by (1.11) is an equilibrium state of the above equation, since it gives $\tilde{\mu}_0 = 0$. From now on we assume $\tilde{u}_0 = U$ and $\tilde{\mu}_0 = 0$.

3.2. The moderate time scale $t = O(1)$: Equilibration of $\tilde{\mu}_1$. In this time scale, the outer solution satisfies

$$\partial_t u_0 = 0, \quad \partial_t u_1 = 0. \tag{3.5}$$

For the inner solution, since

$$u_t = \tilde{u}_t + \nabla_s \tilde{u}\cdot\frac{\partial s}{\partial t} + \varepsilon^{-1}\frac{\partial \rho}{\partial t}\tilde{u}_z = \varepsilon^{-1}\frac{\partial \rho}{\partial t}\tilde{u}_{0z}\chi_0 + O(1), \tag{3.6}$$

and $\tilde{u}_0 = U, \tilde{\mu}_0 = 0$, matching the $O(\varepsilon^{-1})$ terms of (3.6) and (2.16) gives

$$\frac{\partial \rho}{\partial t}U_z\chi_0 = \chi_0\Phi(\tilde{u})\frac{\partial}{\partial z}\bigg((1-U^2)\tilde{\mu}_{1z}\bigg). \tag{3.7}$$

Since $\tilde{M}(\tilde{u}) = 1 - \tilde{u}^2 = 1 - \tilde{u}_0^2 + \varepsilon \tilde{M}'(\tilde{u}_0)\tilde{u}_1 + O(\varepsilon^2)$, its sign in Z_0 is determined by the leading order term $1 - \tilde{u}_0^2$, which is positive, i.e., $\Phi(\tilde{u}) = 1$ in Z_0. To find $\frac{\partial \rho}{\partial t}$, we integrate (3.7) over $(-\eta, \eta)$ and obtain

$$\frac{\partial \rho}{\partial t} U \Big|_{-\eta}^{\eta} = (1 - U^2)\tilde{\mu}_{1z} \Big|_{-\eta}^{\eta} = O(\varepsilon). \tag{3.8}$$

So $\frac{\partial \rho}{\partial t} = 0$ and the normal velocity of the interface is zero in the leading order. Hence $\tilde{\mu}_1$ is in fact equilibrated in $(-\eta, \eta)$. The equilibrium solution satisfies

$$\frac{\partial}{\partial z}\left((1 - U^2)\tilde{\mu}_{1z}\right) = 0 \quad \text{for } z \in (-\eta, \eta). \tag{3.9}$$

So there is a_1 independent of $z \in (-\eta, \eta)$ such that $(1 - U^2)\tilde{\mu}_{1z} = a_1$. Since $1 - U^2 \to O(\varepsilon)$ as $z \to \pm\eta$ and $\tilde{\mu}_{1z} = a_1(1 - U^2)^{-1}$, the only way for $\tilde{\mu}_{1z}$ to remain $O(1)$ is that $a_1 = 0$. This is also consistent with the behavior of $\tilde{\mu}_1$ as $z \to \pm\infty$, which is $\tilde{\mu}_{1z} \to \partial_{\mathbf{n}} \mu_0^{\pm} = 0$ as $z \to \pm\infty$ by the match condition (2.20). So we expect the equilibrium state of $\tilde{\mu}_1$ to be independent of z. By considering the solvability of (2.7), as is done in [27] or [14], we obtain

$$\tilde{\mu}_1 = -S\kappa_0. \tag{3.10}$$

Again by the match condition (2.20), we have $\mu_1 = -S\kappa_0$ on Γ. This is the Gibbs–Thomson condition.

4. The time scale $t_1 = \varepsilon t$: Porous medium equation. Inside the transition region, we have

$$u_t = \varepsilon\left(\tilde{u}_{t_1} + \nabla_s \tilde{u} \cdot \frac{\partial s}{\partial t_1}\right) + \frac{\partial \rho}{\partial t_1}\tilde{u}_z = \frac{\partial \rho}{\partial t_1}\tilde{u}_{0z}\chi_0 + O(\varepsilon). \tag{4.1}$$

Matching (4.1) and (2.16), the ε^{-2} terms indicate that $\tilde{u}_0 = U$ and $\tilde{\mu}_0 = 0$, and the ε^{-1} terms indicate that $\tilde{\mu}_1 = -S\kappa_0$. Combined with $\Phi(\tilde{u}) = 1$ in Z_0, the $O(1)$ terms can be simplified as

$$\frac{\partial \rho}{\partial t_1}\tilde{u}_{0z}\chi_0 = \chi_0 \frac{\partial}{\partial z}\left((1 - \tilde{u}_0^2)\tilde{\mu}_{2z}\right).$$

Integrating in z from $-\infty$ to ∞, we have

$$\frac{\partial \rho}{\partial t_1} U \Big|_{-\eta}^{\eta} = (1 - U^2)\tilde{\mu}_{2z} \Big|_{-\eta}^{\eta} = O(\varepsilon). \tag{4.2}$$

So $\frac{\partial \rho}{\partial t_1} = 0$ and the normal velocity is 0 in the leading order.

In the outer regions $\Omega_{\pm}$,

$$\partial_t u = \varepsilon^2 \partial_{t_1} u_1 + \varepsilon^3 \partial_{t_1} u_2 + \dots. \tag{4.3}$$

Matching the terms of (4.3) and (2.17) for $O(\varepsilon^2)$, we have

$$\partial_{t_1} u_1 = \nabla \cdot \left(|\tilde{M}'(u_0)u_1| \nabla \mu_1\right). \tag{4.4}$$

Since $u_0 = \pm 1$ in $\Omega_{\pm}$, respectively, and $\mu_1 = F''(u_0)u_1$, it turns out that μ_1 satisfies the following porous medium equation:

$$\partial_{t_1} \mu_1 = \nabla \cdot (|\tilde{M}'(u_0)\mu_1| \nabla \mu_1) \quad \text{in } \Omega_{\pm}, \tag{4.5}$$

$$\mu_1 = -S\kappa_0 \quad \text{on } \Gamma. \tag{4.6}$$

5. The time scale $t_2 = \varepsilon^2 t$: Surface diffusion and porous medium diffusion. For the data in the outer regions $\Omega_\pm$, in this time scale, since

$$\partial_t u = \varepsilon^3 \partial_{t_2} u_1 + O(\varepsilon^4), \tag{5.1}$$

by matching the terms of (5.1) and (2.17) for every order in ε, the first nontrivial order is $O(\varepsilon^2)$, which can be simplified as

$$0 = \nabla \cdot \left(|\tilde{M}'(u_0)\mu_1| \nabla \mu_1 \right) \quad \text{in } \Omega_\pm. \tag{5.2}$$

This is a quasi-stationary porous medium equation for μ_1 in Ω_+ and Ω_-, respectively.

In the transition region Γ_ε, we have

$$u_t = \varepsilon^2 \left(\tilde{u}_{t_2} + \nabla_s \tilde{u} \cdot \frac{\partial s}{\partial t_2} \right) + \varepsilon \frac{\partial \rho}{\partial t_2} \tilde{u}_z = \varepsilon \frac{\partial \rho}{\partial t_2} \tilde{u}_{0z} \chi_0 + O(\varepsilon^2). \tag{5.3}$$

By (1.1), matching (5.3) and (2.16), the ε^{-2}, ε^{-1} terms indicate that

$$\tilde{u}_0 = U, \ \tilde{\mu}_0 = 0, \ \tilde{\mu}_1 = -S\kappa_0.$$

Under these conditions, the ε^0 term gives

$$\chi_0 \frac{\partial}{\partial z} \left((1 - \tilde{u}_0^2) \tilde{\mu}_{2z} \right) = 0. \tag{5.4}$$

Hence $\tilde{\mu}_2$ is independent of z in $(-\eta, \eta)$. By match condition (2.21), we see that $\lim_{z \to \pm\infty} \mu_{2z} = \partial_{\mathbf{n}} \mu_1^\pm$; that is, μ_2 is asymptotically linear as $z \to \pm\infty$. As for the precise behavior of $\tilde{\mu}_2$ for $|z| > \eta$, we will extrapolate a bit so that $\tilde{\mu}_2$ is constant for $z \in (-\hat{\eta}, \hat{\eta})$. Here we assume $\hat{\eta} > \eta$ so that $1 - \tilde{u}^2$ has definite signs matching that of $1 - u^2$ in the outer regions; that is, $\Phi(\tilde{u}) = \operatorname{sgn}(1 - \tilde{u}^2) = (\operatorname{sgn}(1 - u^2))^\pm$ in $(\hat{\eta}, \infty)$ and $(-\infty, -\hat{\eta})$.

The $O(\varepsilon)$ terms can be simplified as

$$\begin{aligned} \frac{\partial \rho}{\partial t_2} \tilde{u}_{0z} \chi_0 = \Phi(\tilde{u}) \Bigg\{ & \chi_0 (1 - \tilde{u}_0^2) \Delta_s \tilde{\mu}_1 + \chi_0 \frac{\partial}{\partial z} \left((1 - \tilde{u}_0^2) \tilde{\mu}_{3z} \right) + \tilde{M}'(\tilde{u}_0) \frac{\partial}{\partial z} \left(\tilde{u}_1 \tilde{\mu}_{2z} \right) \\ & + \chi_1 \frac{\partial}{\partial z} \left(\varepsilon^{-1} (1 - \tilde{u}_0^2) \tilde{\mu}_{2z} \right) \Bigg\}. \end{aligned} \tag{5.5}$$

Integrating (5.5) from $-\infty$ to $+\infty$, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t_2} U \Big|_{-\eta}^{\eta} = & \left(\int_{-\eta}^{\eta} (1 - U^2) \, dz \right) \Delta_s \tilde{\mu}_1 + \int_{-\infty}^{\infty} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \frac{\partial}{\partial z} (\tilde{u}_1 \tilde{\mu}_{2z}) \, dz \\ & + (1 - U^2) \tilde{\mu}_{3z} \Big|_{-\eta}^{\eta} + \varepsilon^{-1} (1 - U^2) \tilde{\mu}_{2z} \Big|_{\eta}^{2\eta} + \varepsilon^{-1} (1 - U^2) \tilde{\mu}_{2z} \Big|_{-2\eta}^{-\eta}. \end{aligned} \tag{5.6}$$

To calculate $\int_{-\infty}^{\infty} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \frac{\partial}{\partial z} (\tilde{u}_1 \tilde{\mu}_{2z}) \, dz$, since $\tilde{\mu}_{2z} = 0$ in $(-\hat{\eta}, \hat{\eta})$, and $\tilde{u}_{0z} = U_z$ decays exponentially to 0 for $|z| > \hat{\eta}$, and $|\tilde{u}_{0z}(\pm\hat{\eta})| \le O(\varepsilon)$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \frac{\partial}{\partial z} (\tilde{u}_1 \tilde{\mu}_{2z}) \, dz = \int_{|z| > \hat{\eta}} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \frac{\partial}{\partial z} (\tilde{u}_1 \tilde{\mu}_{2z}) \, dz \\ & = \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \tilde{u}_1 \tilde{\mu}_{2z} \Big|_{\hat{\eta}}^{\infty} + \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \tilde{u}_1 \tilde{\mu}_{2z} \Big|_{-\infty}^{-\hat{\eta}} - \int_{|z| > \hat{\eta}} \Phi(\tilde{u}) \tilde{M}''(\tilde{u}_0) \tilde{u}_{0z} \tilde{u}_1 \tilde{\mu}_{2z} \, dz \\ & = \left(\lim_{z \to +\infty} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \tilde{u}_1 \tilde{\mu}_{2z} - \lim_{z \to -\infty} \Phi(\tilde{u}) \tilde{M}'(\tilde{u}_0) \tilde{u}_1 \tilde{\mu}_{2z} \right) + O(\varepsilon). \end{aligned}$$

So, after simplification, (5.6) becomes

$$2\frac{\partial \rho}{\partial t_2} = -2\sigma\Delta_s\kappa_0 + \left(\lim_{z\to+\infty}\Phi(\tilde{u})\tilde{M}'(\tilde{u}_0)\tilde{u}_1\tilde{\mu}_{2z} - \lim_{z\to-\infty}\Phi(\tilde{u})\tilde{M}'(\tilde{u}_0)\tilde{u}_1\tilde{\mu}_{2z}\right) + O(\varepsilon).$$

Here σ is as given in (1.12). Since

$$1-u^2 = 1-u_0^2+\varepsilon\tilde{M}'(u_0)u_1 + O(\varepsilon^2) = \varepsilon\tilde{M}'(u_0)u_1 + O(\varepsilon^2)$$

and its sign is determined by the leading order term, we have $\operatorname{sgn}(1-u^2) = -\operatorname{sgn}(u_0u_1)$ as long as $u_0u_1 \neq 0$. So

$$\lim_{z\to+\infty}\Phi(\tilde{u})\tilde{M}'(\tilde{u}_0)\tilde{u}_1\tilde{\mu}_{2z} = \left(\operatorname{sgn}(1-u^2)\tilde{M}'(u_0)u_1\partial_{\mathbf{n}}\mu_1\right)^+ = -|\tilde{M}'(u_0)u_1^+|\partial_{\mathbf{n}}\mu_1^+,$$

$$\lim_{z\to-\infty}\Phi(\tilde{u})\tilde{M}'(\tilde{u}_0)\tilde{u}_1\tilde{\mu}_{2z} = \left(\operatorname{sgn}(1-u^2)\tilde{M}'(u_0)u_1\partial_{\mathbf{n}}\mu_1\right)^- = -|\tilde{M}'(u_0)u_1^-|\partial_{\mathbf{n}}\mu_1^-,$$

and

$$\frac{\partial \rho}{\partial t_2} = -\sigma\Delta_s\kappa_0 + \frac{1}{2}\left[|\tilde{M}'(u_0)u_1|\partial_{\mathbf{n}}\mu_1\right]_-^+ + O(\varepsilon).$$

Neglecting lower order terms, the normal velocity $V_{\mathbf{n}}$ is

$$V_{\mathbf{n}} = -\frac{\partial \rho}{\partial t_2} = \sigma\Delta_s\kappa_0 - \frac{1}{2}\left[|\tilde{M}'(u_0)u_1|\partial_{\mathbf{n}}\mu_1\right]_-^+. \tag{5.7}$$

Combining (5.2) and (5.7), and by $\mu_1 = F''(u_0)u_1$, we have the following evolution law of the interface:

$$\nabla\cdot(|\mu_1|\nabla\mu_1) = 0 \qquad \text{in } \Omega_\pm, \tag{5.8}$$

$$\mu_1 = -S\kappa_0 \qquad \text{on } \Gamma, \tag{5.9}$$

$$V_{\mathbf{n}} = \sigma\Delta_s\kappa_0 - \left[\frac{|\tilde{M}'(u_0)\mu_1|}{2F''(u_0)}\partial_{\mathbf{n}}\mu_1\right]_-^+ \qquad \text{on } \Gamma. \tag{5.10}$$

6. Discussion. Cahn–Hilliard equations with various diffusion mobility have profound connections to interfacial motions which have been studied extensively from various perspectives, including modeling and analysis [7, 10, 16, 24, 31], numerical simulations [2, 11, 12, 28, 29, 33], and experimental comparisons in physical systems [23, 30]. Given the multiscale nature of the underlying complex microstructure evolution, there have been many challenges in performing quantitative experimental measurements and long time numerical simulations. Several observations have been debated in recent years, including the pinning effects in spinodal decomposition of polymer mixtures [2, 11, 23, 30]. The different roles played by diffusion in bulk phases and the surface diffusion on different spatial and temporal scales due to the disparate diffusional mobilities are important factors contributing to the complexity [5, 14, 21, 25, 28].

While some past works have explored the characteristics of the interfacial motion described by the Cahn–Hilliard equation, we make a new attempt to analytically understand the dependence of the associated interfacial dynamics on the specific forms of the nonlinear potential and the nonlinear and degenerate diffusion mobility. Our

results show that the interface motion depends crucially on the degeneracy of the diffusion mobility. As opposed to the generally believed scenario that the corresponding surface motion is purely determined by surface diffusion, we show the existence of a quasi-stationary porous medium diffusion process in the bulk phases, together with surface diffusion. This explains the coarsening mechanism for such systems, especially the off-critical situations.

Although our studies are mostly of a mathematical nature and are derived for a special scalar Cahn–Hilliard equation only, they may have a broader impact on phase field modeling of realistic physical, material, or biological systems. For example, it is often of much interest to explore how different phase field models correspond to various scenarios of microstructure evolution in which either bulk and surface diffusion coexist or one dominates the other. Experimental testing of the theories describing the microstructure coarsening is often quite difficult to perform so that analysis of relevant mathematical models may help us gain further insight into corresponding physical processes. Our theoretical findings can also be used as benchmarks for numerical simulations. Many issues remain to be explored. In the future, we plan to study similar problems for more general models, such as those incorporating anisotropic elastic effect. Meanwhile, it will also be interesting to pursue a more rigorous mathematical study on the well-posedness and stability of the newly derived interfacial dynamic laws, such as those represented by (1.13)–(1.15). We will also consider performing additional large scale numerical simulations to offer more quantitative information on the multiscale dynamics of microstructure evolution.

Appendix.

Proof of Lemma 1.6. (1) For the existence, since $g \geq 0$ on $\partial\omega$, standard theory for harmonic functions and maximum principle show that there exists a unique nonnegative function $\phi \in C^2(\omega) \cap C(\bar{\omega})$ satisfying

$$\Delta\phi = 0 \qquad \text{in } \omega, \tag{A.1}$$

$$\phi = g^2 \geq 0 \qquad \text{on } \partial\omega. \tag{A.2}$$

Then $\mu = \sqrt{\phi}$ is a nonnegative solution for (1.17)–(1.18). Furthermore, if $g > 0$ somewhere on $\partial\omega$, the strong maximum principal for harmonic functions guarantees that $\phi > 0$ in ω and hence $\mu > 0$ in ω.

(2) For the uniqueness, suppose μ_1 is any $C^2(\omega) \cap C(\bar{\omega})$ solution for (1.17)–(1.18). We need to show that $\mu_1 \geq 0$ and $\mu_1 = \sqrt{\phi}$. First, we show that μ_1 cannot be negative anywhere in ω. If not, then there is a connected open subset $\hat{\omega} \subset \omega$, where $\mu_1 < 0$ in $\hat{\omega}$ and $\mu_1 = 0$ on $\partial\hat{\omega}$. Taking $\phi_1 = \mu_1^2$, then $\Delta\phi_1 = 0$ in $\hat{\omega}$, and $\phi_1 = 0$ on $\partial\hat{\omega}$. By the maximum principle, $\phi_1 = 0$ in $\hat{\omega}$, and hence $\mu_1 = 0$ in $\hat{\omega}$, which is a contradiction.

So $\mu_1 \geq 0$ in ω. The absolute value in (1.17) can then be removed, and we have $\nabla \cdot (\mu_1 \nabla \mu_1) = 0$ in ω. Then μ_1^2 solves (A.1)–(A.2). Hence $\mu_1^2 = \phi$, i.e., $\mu_1 = \sqrt{\phi}$. ∎

Acknowledgment. The authors would like to thank the referees for their careful reading and their valuable comments, which improved our work.

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