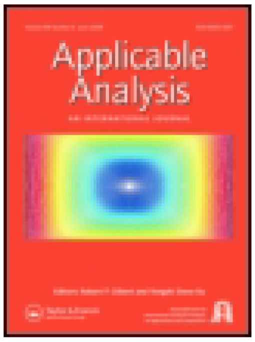

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Weak solutions for the functionalized Cahn–Hilliard equation with degenerate mobility

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ABSTRACT

The functionalized Cahn–Hilliard free energy describes phase separation in mixtures of amphiphilic molecules in solvent. Applications to highly amphiphilic molecules such as lipids requires degenerate diffusion that eliminates bulk diffusion, resulting in surface driven diffusion. We study the existence of weak solutions of a gradient flow of the functionalized Cahn–Hilliard equation that incorporates degenerate mobility, capturing solutions as limits of the weak solution for equations with non-degenerate mobilities.

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1. Introduction

Let $\Omega = [0, 2\pi]^n$ and $T > 0$ be an a given positive number. We consider the gradient flow of the functionalized Cahn–Hilliard (FCH) equation with degenerate mobility,

$$\partial_t u = \operatorname{div}(M(u)\nabla\mu), \qquad (x,t) \in \Omega_T = \Omega \times (0,T), \tag{1}$$

$$\mu = -\Delta\omega + W''(u)\omega - \eta\omega, \tag{2}$$

$$\omega = -\Delta u + W'(u), \tag{3}$$

under the imposition of period boundary condition. The mobility $M(u)$ and double-well potential $W(u)$ are both smooth functions from $\mathbb{R}$ to $\mathbb{R}$. The FCH free energy was derived from models for interfacial energy in phase separated mixtures with an amphiphilic structure by M. Teubner, R. Strey [1], G. Gompper, M. Schick [2], and was developed to describe nanoscale morphology changes in functionalized polymer chains by K. Promislow and B. Wetton[3]. Generally, u represents the volume fraction of amphiphilic material verses solvent, with $u = -1$ denoting pure solvent and $u = 1$ denoting pure amphile. The double-well potential W has local minima at $u = \pm 1$, with $W(-1) = 0 > W(1)$, while the degenerated mobility M is taken to be zero in the $u = -1$ solvent phase.

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The FCH free energy can be understood informally from the following process. Denote the Cahn–Hilliard(CH) free energy

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\,\mathrm{d}x, \tag{4}$$

which was introduced by J. W. Cahn and J. E. Hilliard[4] to describe a binary mixture by a phase field function u. The critical points of E are the zeros of its variational derivative, ω,

$$\omega := \frac{\delta E(u)}{\delta u} = -\Delta u + W'(u). \tag{5}$$

The FCH free energy uses the square of the CH variational derivative to measure distance to criticality, and subtracts terms that reward surface area.

$$F(u) = \int_\Omega \frac{1}{2}\left(\frac{\delta E(u)}{\delta u}\right)^2 - \eta\left(\frac{1}{2}|\nabla u|^2 + W(u)\right)\,\mathrm{d}x, \tag{6}$$

where η is a positive constant. The minus sign in front of η is of considerable significance: it incorporates the propensity of the amphiphilic surfactant phase to drive the creation of interface. Under this construction, the minimizers of the FCH free energy are approximate critical points of the CH, with large surface area. The variational derivative of (6), acting on a test function $\phi \in \mathcal{C}^\infty_{\mathrm{per}}(\Omega)$ takes the form

$$\left(\frac{\delta F(u)}{\delta u}, \phi\right) = \int_\Omega \left(-\Delta u + W'(u)\right)\left(-\Delta\varphi + W''(u)\phi\right) - \eta(-\Delta u + W'(u))\phi\,\mathrm{d}x.$$

Equivalently we may write

$$\mu := \frac{\delta F(u)}{\delta u} = -\Delta\omega + W''(u)\omega - \eta\omega, \tag{7}$$

where μ is called the chemical potential.

The minimizers of the CH free energy are generically single-layers morphologies familiar from the phase separation of two immiscible fluids, such as oil and water, whose components form subdomains that are pure in each component. Conversely the minimizers of the FCH free energy are bilayer morphologies which separate the majority component into regions bounded by a thin layer of other phase – like a soap bubble or surfactant. Significantly, bilayers can rupture, re-uniting the two regions of majority phase, as when a lipid bilayer opens a pore, or tears. The well-posedness of the minimization problem for the energy functional (6) with a more generalized form, including the existence of global minimizers was established in [5, 6] over various natural function spaces. The problem, such as existence of bilayer, pearled patterns, and network bifurcations, were discussed in [7–11], as well as the review article [12].

Under uniform mobility, $M \equiv 1$, the system (1)–(3) is the usual H^{-1} gradient flow associated with the FCH free energy. In this case the competitive evolution of bilayer and pore structures of the FCH equation on a variety of time scales by a multi-scale analysis were studied in [13–15], and sharp interface limits for the dynamics of bilayer structures were derived. The center-unstable spectra and resolvent estimates to the operators associated with gradient flows of the functionalized energies are obtained in [16]. The global-in-time Gevrey regularity solutions for the FCH equation were recently established, see [17]. Numerical methods for the simulation of the uniform mobility system have been presented. including the spectral-Galerkin methods[18], an implicit-explicit scheme and the spectral method[19], and local discontinuous Galerkin method[20].

It is reasonable for the mobility to depend upon the concentration, and in particular for it to effectively vanish in the solvent phase. This would eliminate exchange of amphiphilic molecules between disjoint morphologies. Such examples have been described experimentally, [21], however to the best of our knowledge there are no results for theoretical analysis of the FCH equation with degenerate mobility. The existences of the solutions for the CH equations with a degenerate mobility were established in [22] in one space dimension and in [23, 24] for any dimensions. The main idea of this paper follows the presentation of [24]. We refer [25–27] for the other forms of higher order CH equations.

This paper is organized as follows. In Section 2, we prove the existence of weak solution with positive mobilities for the FCH equation in any dimension by the Galerkin method. In Section 3, we show the existence of more regular weak solutions for the equation (1)–(3) with positive mobilities in the dimension $n \leq 3$. Finally, in Section 4 we obtain the existence of the weak solution for the FCH equation with degenerated mobilities for the dimension $n \leq 3$.

2. Weak solution for the cases with positive mobilities

In this section, we assume the mobility $M(u)$ is a non-degenerate continuous function $M_\theta(u)$

$$M_\theta(u) = \begin{cases} |1+u|^m, & \text{if } |1+u| > \theta, \\ \theta^m, & \text{if } |1+u| \leq \theta, \end{cases} \tag{8}$$

with $0 < m < +\infty$ if $n = 1, 2, 3, 4$ and $0 < m < 4/(n-4)$ if $n \geq 5$. At the same time, we also assume $W(u) \in C^2(\mathbb{R}, \mathbb{R})$ and for all $u \in \mathbb{R}$

$$C_1|u|^{2p} - C_2 \leq W(u) \leq C_3|u|^{2p} + C_4, \tag{9}$$

$$|W'(u)| \leq C_3|u|^{2p-1} + C_4, \tag{10}$$

$$C_1|u|^{2p-2} - C_2 \leq W''(u) \leq C_3|u|^{2p-2} + C_4, \tag{11}$$

$$pC_3|u|^{2p} - C_2 \leq W'(u)u, \quad \text{when } |u| > 1, \tag{12}$$

where $1 < p < \infty$, if $n = 1$, 2 and $1 < p \leq (n-1)/(n-2)$, if $n \geq 3$, $C_i > 0$, $i = 1, \ldots, 4$ are constants. The tilted double-well potential energy

$$W(u; \tau) = |u+1|^2 \left(\frac{1}{2}|u-1|^2 - \frac{\tau}{3}(u-2) \right),$$

is a typical example, see [11].

Theorem 2.1: *Let $u_0(x) \in H^2(\Omega)$. Under the assumptions of (8) and (9)–(12), for any $T > 0$ is a given constant, there exists a pair of functions (u_θ, μ_θ) that satisfies the following conditions*

(i) $u_\theta \in L^\infty(0, T; H^2(\Omega)) \cap C(0, T; H^{2-\varepsilon}(\Omega)) \cap C([0, T]; X)$, *for any* $0 < \varepsilon < 1$, *and* $X = C^\alpha(\Omega)$, $0 < \alpha < \frac{1}{2}$ *if* $n = 1, 2, 3$, $X = L^q(\Omega)$ *with* $1 \leq q < 2n/(n-4)$ *if* $n \geq 4$;
(ii) $\partial_t u_\theta \in L^2(0, T; (H^2(\Omega))')$;
(iii) $u(0) = u_0$;
(iv) $\mu_\theta \in L^2(0, T; H^1(\Omega))$;
(v) *For all* $\xi \in L^2(0, T; H^2(\Omega))$ *and* $\phi \in H^2(\Omega)$, *the following integral equalities hold:*

$$\int_0^T < \partial_t u_\theta, \xi >_{(H^2(\Omega)', H^2(\Omega))} \, \mathrm{d}t + \int_0^T \int_\Omega M_\theta(u_\theta) \nabla \mu_\theta \nabla \xi \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{13}$$

and

$$\int_\Omega \mu_\theta \phi \, \mathrm{d}x = \int_\Omega \left(-\Delta u_\theta + W'(u_\theta)\right) \left(-\Delta \phi + W''(u_\theta)\phi\right) + \eta \left(\Delta u_\theta - W'(u_\theta)\right) \phi \, \mathrm{d}x. \tag{14}$$

In additional, we have

$$\|u_\theta\|_{L^\infty(0,T;H^2(\Omega))} + \int_0^T \int_\Omega M_\theta(u_\theta)|\nabla \mu_\theta|^2 \,\mathrm{d}x\,\mathrm{d}t + \|\omega_\theta\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u_\theta\|_{L^2(0,T;(H^2(\Omega))')} \le C, \tag{15}$$

where C only depends on Ω, $\|u_0\|_{H^2(\Omega)}$, C_i, $i = 1, \dots, 4$ *and* η.

2.1. Galerkin approximation

Let $\mathbb{Z}_+$ be the set of nonnegative integers and $\{\phi_j\}_{j=1}^\infty$ be the complete orthogonal basis for $H^k(\Omega)$, $k \ge 0$ as follows

$$\{(2\pi)^{-n/2}, \quad \mathrm{Re}(\pi^{-n/2}\mathrm{e}^{iK\cdot x}), \quad \mathrm{Im}(\pi^{-n/2}\mathrm{e}^{iK\cdot x}) : K \in \mathbb{Z}_+^n/\{(0,0,\dots,0)\}.$$

We shall find the solution of the following approximating system for the problem (1)–(3),

$$\int_\Omega \partial_t u^N \phi_j \,\mathrm{d}x = -\int_\Omega M_\theta(u^N)\nabla\mu^N \nabla\phi_j \,\mathrm{d}x, \tag{16}$$

$$\int_\Omega \mu^N \phi_j \,\mathrm{d}x = \int_\Omega -\omega^N \Delta\phi_j + W''(u^N)\omega^N \phi_j - \eta\omega^N\phi_j \,\mathrm{d}x, \tag{17}$$

$$u_0^N(x) = \sum_{j=1}^N \left(\int_\Omega u_0\phi_j \,\mathrm{d}x\right)\phi_j(x), \tag{18}$$

where u^N and μ^N are the Galerkin approximations with the form

$$u^N(x,t) = \sum_{j=1}^N c_j^N(t)\phi_j(x), \qquad \mu^N(x,t) = \sum_{j=1}^N d_j^N(t)\phi_j(x),$$

and

$$\omega^N(x) = -\Delta u^N + W'(u^N). \tag{19}$$

Since the right hand side of (16)–(18) satisfies the Carathéodory condition, the existence of a local solution is ensured. The global existence then follows from the following Lemma 2.2 and Lemma 2.3, which gives uniform bounds on $c_j^N(t)$ and $d_j^N(t)$, for $1 \le j \le N$ on the interval of local existence time and allows to prolong the local solution on the whole interval $(0, T)$.

2.2. Some prior estimates

Lemma 2.2: *Let u^N be the solution of (16)–(18). For all $t \in (0, T]$, we have*

$$\int_\Omega u^N \,\mathrm{d}x = \int_\Omega u_0^N \,\mathrm{d}x. \tag{20}$$

Proof: Taking $\phi_j = (2\pi)^{-n/2}$ in (16), it is easy to obtain (20). ∎

Lemma 2.3: *Let u^N be the solution of (16)–(18). We have*

$$\|u^N\|_{L^\infty(0,T;H^2(\Omega))} \le C, \tag{21}$$

$$\int_0^T \int_\Omega M_\theta(u^N)|\nabla\mu^N|^2 \,\mathrm{d}x\,\mathrm{d}t \le C, \tag{22}$$

$$\|\omega^N\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{23}$$

where C only depends on Ω, $\|u_0\|_{H^2(\Omega)}$, C_i, $i = 1, \ldots, 4$ and η.

Proof: Replacing μ^N in place of ϕ_j in equation (16), we have

$$\int_\Omega \partial_t u^N \mu^N \,\mathrm{d}x + \int_\Omega M_\theta(u^N)|\nabla \mu^N|^2 \,\mathrm{d}x = 0. \tag{24}$$

Taking $\partial_t u^N$ in the place of ϕ_j in equation (17) and by (19), we obtain

$$\begin{aligned}\int_\Omega \partial_t u^N \mu^N \,\mathrm{d}x &= \int_\Omega -\omega^N \frac{\partial \Delta u^N}{\partial t} + \omega^N \frac{\partial W'(u^N)}{\partial t} - \eta \omega^N \frac{\partial u^N}{\partial t} \,\mathrm{d}x \\ &= \int_\Omega \omega^N \frac{\partial}{\partial t}(-\Delta u^N + W'(u^N)) - \eta(-\Delta u^N + W'(u^N))\frac{\partial u^N}{\partial t} \,\mathrm{d}x \\ &= \frac{d}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\omega^N|^2 - \eta\left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x. \end{aligned} \tag{25}$$

By (24) and (25), we have

$$\frac{d}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\omega^N|^2 - \eta\left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x + \int_\Omega M_\theta(u^N)|\nabla \mu^N|^2 \,\mathrm{d}x = 0.$$

Integrating with respect to t from 0 to T, we have

$$\begin{aligned}&\int_\Omega \frac{1}{2}|\omega^N|^2 - \eta\left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x + \int_0^T \int_\Omega M_\theta(u^N)|\nabla \mu^N|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \int_\Omega \frac{1}{2}|\omega_0^N|^2 \,\mathrm{d}x - \eta\left(\frac{1}{2}|\nabla u_0^N|^2 + W(u_0^N)\right) \mathrm{d}x \\ &\quad \le C\|u_0^N\|_{H^2}^2. \end{aligned} \tag{26}$$

Let us consider the first term of the left hand of (26). By (12) and (19), we have

$$\begin{aligned}&\int_\Omega \frac{1}{2}|\omega^N|^2 - \eta\left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x \\ &= \int_\Omega \left(\frac{1}{2}|\omega^N|^2 - \eta\omega^N u^N\right) \mathrm{d}x + \int_\Omega \eta\omega^N u^N - \eta\left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x \\ &\ge \int_\Omega \frac{1}{2}|\omega^N|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{4}|\omega^N|^2 \,\mathrm{d}x - \int_\Omega \eta^2|u^N|^2 \,\mathrm{d}x \\ &\quad + \eta\int_\Omega |\nabla u^N|^2 + W'(u^N)u^N - \left(\frac{1}{2}|\nabla u^N|^2 + W(u^N)\right) \mathrm{d}x \\ &\ge \int_\Omega \frac{1}{4}|\omega^N|^2 + \frac{\eta}{2}|\nabla u^N|^2 \,\mathrm{d}x + \eta\int_\Omega (W'(u^N)u^N - W(u^N) - \eta|u^N|^2) \,\mathrm{d}x \\ &\ge \int_\Omega \frac{1}{4}|\omega^N|^2 + \frac{\eta}{2}|\nabla u^N|^2 \,\mathrm{d}x + C_3(p-1)\eta\int_\Omega |u^N|^{2p} \,\mathrm{d}x - \eta\int_\Omega \eta|u^N|^2 \,\mathrm{d}x. \end{aligned}$$

Since $p > 1$, we have

$$\int_\Omega \left(|\omega^N|^2 + |\nabla u^N|^2 + \int_\Omega |u^N|^{2p}\right) \mathrm{d}x + \int_0^T \int_\Omega M_\theta(u^N)|\nabla \mu^N|^2 \,\mathrm{d}x\,\mathrm{d}t \le C,$$

where C depends only on $\|u_0\|_{H^2(\Omega)}$, C_i, $i = 1, \ldots, 4$ and η. The estimates of (22) and (23) are established.

Since

$$\int_\Omega |\omega^N|^2 \,\mathrm{d}x = \int_\Omega |-\Delta u^N + W'(u^N)|^2 \,\mathrm{d}x \le C$$

and $H^1(\Omega) \subset L^{4p-2}(\Omega)$ for $1 < p < \infty$ if $n=1, 2$ or $1 < p \le \frac{n-1}{n-2}$ if $n \ge 3$ is a continuous embedding, we get

$$\begin{aligned}\int_\Omega |-\Delta u^N|^2 \,\mathrm{d}x \le & 2\left(\int_\Omega |W'(u^N)|^2 \,\mathrm{d}x + \int_\Omega |\omega^N|^2 \,\mathrm{d}x\right)\\ \le & C\int_\Omega |u^N|^{4p-2} \,\mathrm{d}x + C\\ \le & C\int_\Omega |\nabla u^N|^2 \,\mathrm{d}x + C.\end{aligned} \tag{27}$$

So the estimate (21) is obtained by (20) and (27). ■

The Sobolev embedding theorem for the space $H^2(\Omega)$ yields the following corollaries.

Corollary 2.4: *Let u^N be the solution of (16)–(18). We have*

$$\|u^N\|_{L^\infty(0,T;X)} \le C, \quad \text{if } n = 1, 2, 3, \tag{28}$$

where $X = C^\alpha(\Omega)$ with $0 < \alpha < \frac{1}{2}$ for $n=1, 2, 3$, and $X = L^q(\Omega)$, $1 \le q < \infty$ for $n=4$, $1 \le q \le 2n/(n-4)$ for $n > 4$.

Corollary 2.5: *Let u^N be the solution of (16)–(18). We have*

$$\|M_\theta(u^N)\|_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad \text{if } n = 1, 2, 3, \tag{29}$$

and

$$\|M_\theta(u^N)\|_{L^\infty(0,T;X)} + \|\sqrt{M_\theta(u^N)}\|_{L^\infty(0,T;Y)} \le C, \quad \text{if } n \ge 4, \tag{30}$$

where $X = L^{q_1}(\Omega)$, $Y = L^{q_2}(\Omega)$ with $1 \le q_1, q_2 < \infty$, if $n=4$, and $1 \le q_1 \le 2n/m(n-4)$, $1 \le q_2 \le 4n/m(n-4)$ if $n \ge 5$.

Corollary 2.6: *Let u^N be the solution of (16)–(18). We have*

$$\|W'(u^N)\|_{L^\infty(0,T;X)} + \|W''(u^N)\|_{L^\infty(0,T;Y)} \le C, \tag{31}$$

where $X = Y = L^\infty(\Omega)$ if $n=1, 2, 3$, and $X = L^{q_1}(\Omega)$, $Y = L^{q_2}(\Omega)$ with $1 \le q_1, q_2 < \infty$, if $n=4$, and $1 \le q_1 \le (1/(2p-1))(2n/(n-4))$, $1 \le q_2 \le (1/(p-1))(n/(n-4))$, if $n > 4$.

Lemma 2.7: *Let u^N be the solution of (16)–(18). We have*

$$\|\mu^N\|_{L^2(0,T;H^1(\Omega))} \le C_\theta, \tag{32}$$

where C depend on θ, Ω, $\|u_0\|_{H^2(\Omega)}$, C_i, $i = 1, \ldots, 4$ and η.

Proof: Taking $\phi_j = (2\pi)^{-n/2}$ in (17), and by Corollary 2.6, we have

$$\begin{aligned}\int_\Omega \mu^N \,\mathrm{d}x &= \int_\Omega W''(u^N)\omega^N - \eta\omega^N \,\mathrm{d}x \\ &\le \int_\Omega |W''(u^N)|^2 \,\mathrm{d}x + C\int_\Omega |\omega^N|^2 \,\mathrm{d}x + C \\ &\le C.\end{aligned}$$

By (8), we have $M_\theta(u) \ge \theta^m$. So by (22), we have

$$\int_0^T \int_\Omega |\nabla\mu^N|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\theta^{-m}.$$

We get (32) by Poincaré inequality. ■

Lemma 2.8: *Let u^N be the solution of (16)–(18). We have*

$$\|\partial_t u^N\|_{L^2(0,T;(H^2(\Omega))')} \le C, \tag{33}$$

where C only depend on Ω, $\|u_0\|_{H^2(\Omega)}$, C_i, $i = 1, \ldots, 4$ and η.

Proof: For any $\phi \in L^2(0,T;H^2(\Omega))$, by Hölder inequality, we have

$$\begin{aligned}\left|\int_\Omega \frac{\partial u^N}{\partial t}\phi \,\mathrm{d}x\right| &= \left|\int_\Omega M_\theta(u^N)\nabla\mu^N\nabla\phi \,\mathrm{d}x\right| \\ &\le \|\sqrt{M_\theta(u^N)}\|_{L^n(\Omega)}\|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega)}\|\nabla\phi\|_{L^{2n/(n-2)}}.\end{aligned}$$

By Corollary 2.5, we get

$$\begin{aligned}\left|\int_0^T \int_\Omega \frac{\partial u^N}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}t\right| &\le C\int_0^T \|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(\Omega)}\|\phi\|_{H^2}\,\mathrm{d}t \\ &\le \|\sqrt{M_\theta(u^N)}\nabla\mu^N\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{L^2(0,T;H^2(\Omega))} \\ &\le C.\end{aligned}$$

The lemma is proved. ■

2.3. Proof of theorem 2.1

By Aubin-Lions Lemma, there exists a subsequence of u^N (not relabeled) and a function u_θ such that as $N \to \infty$

$$u^N \rightharpoonup u_\theta, \quad \text{weak}-* \text{ in } L^\infty(0,T;H^2(\Omega)), \tag{34}$$

$$u^N \to u_\theta, \quad \text{strongly in } C(0,T;H^{2-\varepsilon}(\Omega)) \text{ for any } 0 < \varepsilon < 1, \tag{35}$$

$$u^N \to u_\theta, \quad \text{strongly in } C(0,T;X) \text{ and a.e. in } Q_T, \tag{36}$$

$$\partial_t u^N \rightharpoonup \partial_t u_\theta, \quad \text{weakly in } L^2(0,T;(H^2(\Omega))'), \tag{37}$$

where $X = C^\alpha(\Omega)$ with $0 < \alpha < \frac{1}{2}$ for $n = 1, 2, 3$, and $X = L^q(\Omega)$, $1 \le q < \infty$ if $n = 4$, $q < (2n/(n-4))$ for $n > 4$. In addition, we obtain the following bound for u_θ,

$$\|u_\theta\|_{L^\infty(0,T;H^2(\Omega))} + \|\partial_t u_\theta\|_{L^2(0,T;(H^2(\Omega))')} \le C.$$

By (32), there exist a subsequence of μ^N (not relabeled) and a function μ_θ such that as $N \to \infty$

$$\mu^N \to \mu_\theta, \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{38}$$

By (36), when $n \le 4$, $M_\theta(u^N) \to M_\theta(u_\theta)$, strongly in $C([0,T];L^n(\Omega))$. So we have

$$M_\theta(u^N)\nabla\mu^N \rightharpoonup M_\theta(u_\theta)\nabla\mu_\theta, \quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)). \tag{39}$$

When $n \ge 5$, by (36) again, we have

$$M_\theta(u^N) \to M_\theta(u_\theta), \quad \text{strongly in } C([0,T];L^{n/2}(\Omega)), \tag{40}$$

$$\sqrt{M_\theta(u^N)} \to \sqrt{M_\theta(u_\theta)}, \quad \text{strongly in } C([0,T];L^n(\Omega)). \tag{41}$$

By (22), there exists a subsequence of $\sqrt{M_\theta(u^N)}\nabla\mu^N$ (not relabeled) and a function χ_θ such that

$$\sqrt{M_\theta(u^N)}\nabla\mu^N \rightharpoonup \chi_\theta, \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Combining with (40) and (41), we have

$$\sqrt{M_\theta(u^N)}(\sqrt{M_\theta(u^N)}\nabla\mu^N) \rightharpoonup \sqrt{M_\theta(u_\theta)}\chi_\theta, \quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)),$$
$$M_\theta(u^N)\nabla\mu^N \rightharpoonup M_\theta(u_\theta)\nabla\mu_\theta, \quad \text{weakly in } L^2(0,T;L^{2n/(n+4)}(\Omega)).$$

We also get (39) by the uniqueness of weak limits.

By (37) and (39) and letting $N \to \infty$ in (16), we have

$$\int_0^T < \partial_t u_\theta, \phi >_{(H^2(\Omega)',H^2(\Omega))} \,\mathrm{d}t = -\int_0^T \int_\Omega M_\theta(u_\theta)\nabla\mu_\theta\nabla\phi \,\mathrm{d}x\,\mathrm{d}t,$$

for all $\phi \in L^2(0,T;H^2(\Omega))$.

By (31) and (34), we have

$$\omega^N = -\Delta u^N + W'(u^N) \rightharpoonup \omega_\theta, \quad \text{weakly}-^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{42}$$

where $\omega_\theta = -\Delta u_\theta + W'(u_\theta)$. Then for any $\phi \in H^2(\Omega)$, we get

$$\omega^N\phi \rightharpoonup \omega_\theta\phi, \quad \text{weakly}-^* \text{ in } L^\infty(0,T;L^{n/(n-2)}(\Omega)). \tag{43}$$

By (31), we have

$$W''(u^N) \to W''(u_\theta), \quad \text{strongly in } L^{n/2}(\Omega) \text{ for } t \in (0,T) \text{ a.e..} \tag{44}$$

By (38), (42), (43), and (44) and taking $N \to \infty$ in (17), we have

$$\int_0^T \int_\Omega \mu_\theta\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \left(\omega_\theta(-\Delta\phi + W''(u_\theta)\phi) + \eta\omega_\theta\phi\right) \mathrm{d}x\,\mathrm{d}t$$

for all $\phi \in L^2(0,T;H^2(\Omega))$. Together, these results yields (14).

3. Weak solution with positive mobilities for $n = 1, 2, 3$

In this section, we assume $n = 1, 2, 3$, require the non-degenerate function $M_\theta(u)$ to satisfy (8), and further assume that $W(u) \in C^4(\mathbb{R};\mathbb{R})$ satisfies the condition (9)–(12). Under these additional conditions we obtain smoother solution to the problem (1)–(3).

Theorem 3.1: *Let $u_0(x) \in H^2(\Omega)$. Under the assumptions of (8) and (9)–(12), for any $T > 0$ is a given constant, there exists a functions u_θ, such that it satisfies the following conditions*

(i) $u_\theta \in L^2(0,T;H^5(\Omega)) \cap C(0,T;H^{5-\varepsilon}(\Omega)) \cap C([0,T];C^{3,\alpha}(\Omega))$, $0 < \alpha < 1/2$;
(ii) $\partial_t u_\theta \in L^2(0,T;(H^2(\Omega))')$;
(iii) $u(0) = u_0$;
(iv) *For all $\xi \in L^2(0,T;H^2(\Omega))$, the following integral equality holds:*

$$\int_0^T < \partial_t u_\theta, \xi >_{H^2(\Omega)',H^2(\Omega)} \,\mathrm{d}t = -\int_0^T\int_\Omega M_\theta(u_\theta)\left(-\nabla\Delta\omega_\theta + W'''(u_\theta)\nabla u_\theta\omega_\theta + W''(u_\theta)\nabla\omega_\theta - \eta\nabla\omega_\theta\right)\nabla\xi \,\mathrm{d}x\,\mathrm{d}t, \tag{45}$$

where $\omega_\theta = -\Delta u_\theta + W'(u_\theta)$.

3.1. Further estimates of u_θ

Replacing ϕ_j by ω^N in (17) and integrating in $[0,T]$, we have

$$\int_0^T\int_\Omega \mu^N\omega^N \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega |\nabla\omega^N|^2 + W''(u^N)|\omega^N|^2 - \eta|\omega^N|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

By Young's inequality and Corollary 2.6, we have

$$\int_0^T\int_\Omega |\nabla\omega^N|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_\theta\left(\int_0^T\int_\Omega |\mu^N|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega |\omega^N|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \le C_\theta. \tag{46}$$

By (19), we have

$$\begin{aligned}\int_0^T\int_\Omega |\nabla\Delta u^N|^2 \,\mathrm{d}x\,\mathrm{d}t &\le 2\int_0^T\int_\Omega |\nabla\omega^N|^2 \,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T\int_\Omega |W''(u^N)|^2|\nabla u^N|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C_\theta.\end{aligned} \tag{47}$$

By Sobolev embedding theorem, we could also get

$$\|\nabla u^N\|_{L^2(0,T;C^\alpha(\Omega))} \le \|u^N\|_{L^2(0,T;H^3(\Omega))} \le C_\theta, \tag{48}$$

with $0 < \alpha < 1/2$. By (17) and (19), we have

$$\mu^N = -\Delta\omega^N + W''(u^N)\omega^N - \eta\omega^N,$$

in the sense of distribution. Then by (32) and (28), we could obtain

$$\begin{aligned}\int_0^T\int_\Omega |\Delta\omega^N|^2\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T\int_\Omega |\mu^N|^2 + |W''(u^N)\omega^N|^2\,\mathrm{d}x\,\mathrm{d}t + \eta\int_0^T\int_\Omega |\omega^N|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_\theta.\end{aligned}$$

By Sobolev embedding theorem, we have

$$\|\omega^N\|_{L^2(0,T;L^\infty(\Omega))} \le \|\omega^N\|_{L^2(0,T;H^2(\Omega))} \le C_\theta. \tag{49}$$

Again, by (17) and (19), we get

$$\nabla\mu^N = -\nabla\Delta\omega^N + W'''(u^N)\nabla u^N\omega^N + W''(u^N)\nabla\omega^N - \eta\nabla\omega^N,$$

in the sense of distribution. So by (28), (32), (46), (48) and (49), we could obtain

$$\begin{aligned}\int_0^T\int_\Omega |\nabla\Delta\omega^N|^2\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T\int_\Omega |\nabla\mu^N|^2 + |W'''(u^N)\nabla u^N\omega^N|^2 + |W''(u^N)\nabla\omega^N|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + C\eta^2\int_0^T\int_\Omega |\nabla\omega^N|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_\theta.\end{aligned} \tag{50}$$

By (19), we have

$$\begin{aligned}\nabla\Delta\omega^N = -\nabla\Delta^2u^N + W^{(4)}(u^N)|\nabla u^N|^2\nabla u^N + W'''(u^N)\nabla(|\nabla u^N|^2)\\ + W'''(u^N)\nabla u^N\Delta u^N + W''(u^N)\nabla\Delta u^N.\end{aligned}$$

By (28), (50), (48), and (47), we have

$$\begin{aligned}\int_0^T\int_\Omega |\nabla\Delta^2u^N|^2\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T\int_\Omega |\nabla\Delta\omega^N|^2 + |\nabla u^N|^6 + |\nabla u^N|^2|\nabla^2u^N|^2 + |\nabla\Delta u^N|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_\theta.\end{aligned}$$

3.2. Convergence of $\{u^N\}$

By Aubin-Lions Lemma, there exists a subsequence of u^N (not relabeled) and a function u_θ such that as $N\to\infty$

$$u^N \rightharpoonup u_\theta, \quad \text{weak}-* \text{ in } L^\infty(0,T;H^5(\Omega)), \tag{51}$$

$$u^N \to u_\theta, \quad \text{strongly in } C(0,T;H^{5-\varepsilon}(\Omega)) \text{ for any } 0<\varepsilon<1, \tag{52}$$

$$u^N \to u_\theta, \quad \text{strongly in } C(0,T;C^{3,\alpha}) \text{ and a.e. in } Q_T, \tag{53}$$

$$\partial_t u^N \rightharpoonup \partial_t u_\theta, \quad \text{weakly in } L^2(0,T;(H^2(\Omega))'), \tag{54}$$

where $0 < \alpha < \frac{1}{2}$. In addition, we could obtain the following bound for u_θ,

$$\|u_\theta\|_{L^\infty(0,T;H^5(\Omega))} + \|\partial_t u_\theta\|_{L^2(0,T;(H^2(\Omega))')} \le C. \tag{55}$$

By (53), we have

$$M_\theta(u^N) \to M_\theta(u_\theta), \quad \text{strongly in } C(0,T;L^q(\Omega)), \tag{56}$$

$$W^{(k)}(u^N) \to W^{(k)}(u_\theta), \quad \text{strongly in } C(0,T;L^q(\Omega)), \tag{57}$$

for any $1 \le q < \infty$, and $k = 0, 1, 2, 3, 4$.

By (42) and (50), we have

$$\nabla\Delta\omega^N \rightharpoonup \nabla\Delta\omega_\theta, \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{58}$$

where $\omega_\theta = -\Delta u_\theta + W'(u_\theta)$.

By (42) and (48), we have

$$\nabla u^N \omega^N \rightharpoonup \nabla u_\theta \omega_\theta, \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{59}$$

By (46), we have

$$\nabla\omega^N \rightharpoonup \nabla\omega_\theta, \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{60}$$

As the similar process of proof of theorem 2.1 in subsection 2.3, by (54), and (57)–(60), we could obtain for any $\xi \in L^2(0,T;H^2(\Omega))$,

$$\int_0^T < \partial_t u_\theta, \xi >_{H^2(\Omega)',H^2(\Omega)} \, dt$$

$$= -\int_0^T \int_\Omega M_\theta(u_\theta) \left(-\nabla\Delta\omega_\theta + W'''(u_\theta)\nabla u_\theta\omega_\theta + W''(u_\theta)\nabla\omega_\theta - \eta\nabla\omega_\theta\right) \nabla\xi \, dx \, dt,$$

with

$$\omega_\theta = -\Delta u_\theta + W'(u_\theta).$$

4. Weak solutions with degenerate mobility

In this section, we assume $n = 1, 2, 3$ and consider the degenerate case for problem (1)–(3). We assume the function $M(u) \in C(\mathbb{R}, \mathbb{R}^+)$ and there exist $\delta > 0$ and $c_0 > 0$ such that $M(u) = |1+u|^m$ for $u \in B_\delta(-1) := (-1-\delta, -1+\delta)$, while $M(u) \ge c_0 > 0$ for $u \in \mathbb{R} \setminus B_\delta(-1)$. In addition, there exist $M_1, M_2 > 0$ such that

$$0 \le M(u) \le M_1|u|^m + M_2, \qquad \text{for all } u \in \mathbb{R}.$$

where m can be any positive number $0 < m < \infty$. We assume that $W(u) \in C^4(\mathbb{R}, \mathbb{R})$ satisfies the conditions (9)–(12).

Theorem 4.1: *Let $u_0(x) \in H^2(\Omega)$. Under the above assumptions of $M(u)$ and (9)–(12), for any $T > 0$ is a given constant, there exists a function u, such that it satisfies the following conditions*

(i) $u \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T];H^{2-\varepsilon}(\Omega)) \cap C([0,T];C^\alpha(\Omega))$, *where* $0 < \varepsilon < 2$ *and* $0 < \alpha < 1/2$;

(ii) $\partial_t u_\theta \in L^2(0,T;(H^2(\Omega))')$;

(iii) $u(0) = u_0$;
(iv) *Letting the set* $P := \{(x,t) \in \Omega_T : |1+u| \neq 0\}$, *we have*
  (a) *there exist a set* $B \subset \Omega_T$ *with* $|\Omega_T \setminus B| = 0$ *and a function* $\zeta : \Omega_T \to \mathbb{R}^n$ *satisfying* $\chi_{B\cap P} M(u)\zeta \in L^2(0,T;L^{2n/(n+2)}(\Omega))$ *such that*

$$\int_0^T < \partial_t u, \phi >_{<(H^2(\Omega))', H^2(\Omega)>} \,\mathrm{d}t = -\int_{B\cap P} M(u)\zeta \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t, \tag{61}$$

for all $\phi \in L^2(0,T;H^2(\Omega))$;
  (b) *if* $\nabla\Delta^2 u \in L^q(U)$ *for some open subset* $U \subset \Omega_T$ *and some* $q > 1$, *then we have*

$$\zeta = -\nabla\Delta\omega + W'''(u)\nabla u\omega + W''(u)\nabla\omega - \eta\nabla\omega, \quad \text{in } U.$$

In additional, the following energy inequality is satisfied for all $t \geq 0$.

$$\int_\Omega \left(|\nabla u(x,t)|^2 + W(u(x,t))\right) \mathrm{d}x + \int_{\Omega_t \cap B \cap P} M(u(x,t))|\zeta(x,\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq \int_\Omega |\nabla u_0|^2 + W(u_0)\,\mathrm{d}x.$$

4.1. Proof of theorem 4.1

Taking $\theta_i = \frac{1}{i}, u_i = u^i_{\theta_i}, \mu_i = \mu^i_{\theta_i}$, and $M_i = M_{\theta_i}$. By Lemma 2.3, Lemma 2.8, there exist a subsequence of u_i (not relabeled) and a function u such that

$$u_i \to u \quad \text{strongly in } C([0,T];C^\alpha(\Omega)) \cap C(0,T;H^{2-\varepsilon}(\Omega)) \text{ and a.e. in } \Omega \times (0,T), \tag{62}$$

as $i \to \infty$ for any $0 < \alpha < 1/2$. By the continuity of $M(u)$, we have

$$M_i(u_i) \to M(u) \qquad \text{strongly in } C([0,T];L^{n/2}(\Omega)), \tag{63}$$

$$\sqrt{M_i(u_i)} \to \sqrt{M(u)} \qquad \text{strongly in } C([0,T];L^n(\Omega)). \tag{64}$$

By (22), there exist a subsequence of $\sqrt{M_i(u_i)}\nabla\mu_i$ (not relabeled) and a function ξ such that as $i \to \infty$,

$$\sqrt{M_i(u_i)}\nabla\mu_i \rightharpoonup \xi, \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{65}$$

Then by (63), (65), we have

$$\lim_{i\to\infty} \iint_{\Omega_T} M_i(u_i)\nabla\mu_i\phi \,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} \sqrt{M(u)}\xi\phi \,\mathrm{d}x\,\mathrm{d}t,$$

for any $\phi \in L^2(0,T;L^{2n/(n-2)}(\Omega_T;\mathbb{R}^n))$.

By (62), we could choose a sequence of positive numbers δ_j that monotonically decreases to 0 and by Egorov's theorem, for every $\delta_j > 0$, there exists a subset $B_j \subset \Omega_T$ with $|\Omega_T \setminus B_j| < \delta_j$ such that $u_i \to u$ uniformly in B_j with

$$B_1 \subset B_2 \subset \cdots \subset B_j \subset B_{j+1} \subset \cdots \Omega_T.$$

Define $B = \cup_{j=1}^\infty B_j$, then $|\Omega_T \setminus B| = 0$. Let $P_j = \{(x,t) \in \Omega_T : |1+u| > \delta_j\}$. Then

$$P_1 \subset P_2 \subset \cdots \subset P_j \subset P_{j+1} \subset \cdots \Omega_T.$$

We set $P = \cup_{j=1}^\infty P_j$. For any $\phi \in L^2(0,T;L^{2n/(n-2)}(\Omega_T;\mathbb{R}^n))$, we have

$$\iint_{\Omega_T} M_i(u_i)\nabla\mu_i\phi \,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T\setminus B_j} + \iint_{B_j\cap P_j} + \iint_{B_j\setminus P_j} M_i(u_i)\nabla\mu_i\phi \,\mathrm{d}x\,\mathrm{d}t. \tag{66}$$

For the first term of (66), we have

$$\lim_{j\to\infty}\lim_{i\to\infty}\iint_{\Omega_T\setminus B_j} M_i(u_i)\nabla\mu_i\phi\,\mathrm{d}x\,\mathrm{d}t=\lim_{j\to\infty}\iint_{\Omega_T\setminus B_j}\sqrt{M(u)}\xi\phi\,\mathrm{d}x\,\mathrm{d}t=0. \tag{67}$$

By the uniform convergence of $u_i \to u$ in B_1, there exists an integer N_1 such that for all $i \geq N_1$, we have

$$|1+u_i|>\frac{\delta_1}{2}\quad \text{in } B_1\cap P_1,\qquad |1+u_i|\leq 2\delta_1\quad \text{in } B_1\setminus P_1.$$

By (22), for any $i \geq N_1$, we get

$$\begin{aligned}\left(\frac{\delta_j}{2}\right)^m\iint_{B_1\cap P_1}|\nabla\mu_i|^2\,\mathrm{d}x\,\mathrm{d}t &\leq \iint_{B_1\cap P_1} M_i(u_i)|\nabla\mu_i|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \iint_{\Omega_T} M_i(u_i)|\nabla\mu_i|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C.\end{aligned}$$

So there exists a subsequence $\nabla\mu_{1,k}$ of $\nabla\mu_i$ with $k=1,2,\dots$ and $i\geq N_1$, that weakly converges to some function $\zeta_1\in L^2(B_1\cap P_1)$. At the same time, we write $u_{1,k}$ as the subsequence of u_i correspondingly. By the same process, we have the subsequence $\nabla\mu_{j,k}$ of $\nabla\mu_{j-1,k}$ with $j=2,3,\dots$ and $k=1,2,\dots$, such that $\nabla\mu_{j,k}$ weakly converges to some function $\zeta_j\in L^2(B_j\cap P_j)$. Of course, we write $u_{j,k}$ as the subsequence of $u_{j-1,k}$ correspondingly, such that

$$|1+u_{j,k}|>\frac{\delta_j}{2}\quad \text{in } B_j\cap P_j,\qquad |1+u_{j,k}|\leq 2\delta_j\quad \text{in} B_j\setminus P_j.$$

Since $\{B_j\cap P_j\}_{j=1}^{\infty}$ is an increasing sequence of sets with a limit $B\cap P$, we have $\zeta_j=\zeta_{j-1}$ a.e. in $B_{j-1}\cap P_{j-1}$. So for almost every $x\in B\cap P$, there exists a function $\zeta(x)$ such that $\zeta(x)=\zeta_j(x)$ for almost everywhere $x\in B_j\cap P_j$ and all $j\geq 1$.

Using the standard diagonal argument, we can extract a subsequence such that $\nabla\mu_{k,N_k}\rightharpoonup\zeta$ weakly in $L^2(B_j\cap P_j)$ for any $j\geq 1$ and then by (64)

$$\chi_{B_j\cap P_j}\sqrt{M_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k}\rightharpoonup\chi_{B_j\cap P_j}\sqrt{M(u)}\zeta\quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)),$$

as $k\to\infty$. Here $\chi_{B_j\cap P_j}$ is the characteristic function of $B_j\cap P_j$. However, by (65), we also know $\sqrt{M_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k}\rightharpoonup\xi$ weakly in $L^2(\Omega_T)$. Then we see that $\xi=\sqrt{M(u)}\zeta$ in every $B_j\cap P_j$, and hence $\xi=\sqrt{M(u)}\zeta$ in $B\cap P$. Consequently, by (64) again, we have

$$\chi_{B\cap P}M_{k,N_k}(u_{k,N_k})\nabla\mu_{k,N_k}\rightharpoonup\chi_{B\cap P}M(u)\zeta\quad \text{weakly in } L^2(0,T;L^{2n/(n+2)}(\Omega)). \tag{68}$$

For the third term of (66), we have

$$\begin{aligned}&\lim_{j\to\infty}\lim_{k\to\infty}\left|\iint_{B_j\setminus P_j}M_{k,N_k}(u_{k,N_k})\nabla\mu_{k,N_k}\phi\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leq \lim_{j\to\infty}\lim_{k\to\infty}\sup_{B_j\setminus P_j}\sqrt{M_{k,N_k}(u_{k,N_k})}\left\|\sqrt{M_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k}\right\|_{L^2(\Omega_T)}|\Omega|^{1/n}\|\phi\|_{L^2(0,T;L^{2n/(n-2)}(\Omega))}\\ &\leq C\lim_{j\to\infty}\lim_{k\to\infty}\max\{(2\delta_j)^{m/2},\theta_{k,N_k}^{m/2}\}\\ &=0.\end{aligned} \tag{69}$$

In (66), we take $u_i = u_{k,N_k}$ as the above subsequence and let $k \to \infty$ and then $j \to \infty$. By (67), (68), and (69), we have

$$\lim_{i\to\infty} \iint_{\Omega_T} M_i(u_i)\nabla\mu_i\phi \,\mathrm{d}x\,\mathrm{d}t = \iint_{B\cap P} M(u)\zeta \,\mathrm{d}x\,\mathrm{d}t. \tag{70}$$

At last, let us consider the relation between ζ and u for the case of $n=3$. The cases of $n=1$, 2 are similar. If the interior of $(B_j \cap P_j)^\circ$ is not empty for some j, by

$$\nabla\mu_{k,N_k} = -\nabla\Delta\omega_{k,N_k} + W'''(u_{k,N_k})\nabla u_{k,N_k}\omega_{k,N_k} + W''(u_{k,N_k})\nabla\omega_{k,N_k} - \eta\nabla\omega_{k,N_k}, \tag{71}$$

and (28), (47), (48), and (50), we have

$$\zeta = -\nabla\Delta\omega + W'''(u)\nabla u\omega + W''(u)\nabla\omega - \eta\nabla\omega, \tag{72}$$

as $k \to \infty$. At the same time, since

$$\begin{aligned}\lim_{k\to\infty}\int_0^T\int_\Omega \omega_{k,N_k}\varphi \,\mathrm{d}x\,\mathrm{d}t &= \lim_{k\to\infty}\int_0^T\int_\Omega -\Delta u_{k,N_k} + W'(u_{k,N_k})\varphi \,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega -\Delta u + W'(u)\varphi \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for any $\varphi \in L^2(0,T;H^2(\Omega))$, we have $\omega = -\Delta u + W'(u)$ in the sense of distribution.

If $\nabla\Delta^2 u \in L^q(U)$ for some open set U and $q > 1$, then by (62), we have

$$\begin{aligned}\Delta^2 u &\in L^q(0,T;W^{1,q}(\Omega)),\\ \nabla\Delta u &\in L^q(0,T;W^{2,q}(\Omega)),\\ \Delta u &\in L^q(0,T;W^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\ \nabla u &\in L^q(0,T;W^{4,q}(\Omega)) \cap L^\infty(0,T;L^6(\Omega)),\\ u &\in L^q(0,T;W^{5,q}(\Omega)) \cap C(0,T;C^\alpha(\Omega)),\end{aligned}$$

and furthermore by

$$\nabla\Delta\omega = -\nabla\Delta^2 u + W^{(4)}(u)|\nabla u|^2\nabla u + W'''(u)\nabla(|\nabla u|^2) \tag{73}$$
$$+ W'''(u)\nabla u\Delta u + W''(u)\nabla\Delta u, \tag{74}$$

we have $\nabla\Delta\omega \in L^q(U)$ for $r = \min\{q,2\}$. Since $\omega = -\Delta u + W'(u) \in L^\infty(0,T;L^2(\Omega))$, we have $W'''(u)\nabla u\omega \in L^3(U)$. By (72), $\xi \in L^r(U)$. Hence,

$$\nabla\mu_{k,N_k} \rightharpoonup -\nabla\Delta\omega + W'''(u)\nabla u\omega + W''(u)\nabla\omega - \eta\nabla\omega, \tag{75}$$

weakly in $L^q(U)$ with $\omega = -\Delta u + W'(u)$. We may extend the definition of ζ from $B \cap P \cap U$ into $U \setminus (B \cap P)$ by letting $\zeta = -\nabla\Delta\omega + W'''(u)\nabla u\omega + W''(u)\nabla\omega - \eta\nabla\omega$.

Define

$$\tilde{\Omega}_T := \cup\{U \subset \Omega_T : \nabla\Delta u \in L^p(U) \quad \text{for some } q > 1, q \text{ may depend on } U\}.$$

Then $\tilde{\Omega}_T$ is open and $\zeta = -\nabla\Delta\omega + W'''(u)\nabla u\omega + W''(u)\nabla\omega - \eta\nabla\omega$ in $\tilde{\Omega}_T$. ζ is now defined in $(B \cap P) \cup \tilde{\Omega}_T$. Notice that

$$\Omega_T \setminus ((B \cap P) \cup \tilde{\Omega}_T) \subset (\Omega_T \setminus P) \cup (\Omega_T \setminus B).$$

Since $|\Omega_T \setminus B| = 0$ and $M(u) = 0$ in $\Omega_T \setminus P$, the value of ζ outside of $(B \cap P) \cup \tilde{\Omega}_T$ does not contribute to the integral on the right hand side of (70), so we set $\zeta = 0$ outside of $(B \cap P) \cup \tilde{\Omega}_T$. This completes the proof of Theorem 4.1.

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